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Shifted symplectic Lie algebroids

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Abstract

Shifted symplectic Lie and L_∞ algebroids model formal neighbourhoods of manifolds in shifted symplectic stacks, and serve as target spaces for twisted variants of classical AKSZ topological field theory. In this paper, we classify zero-, one- and two-shifted symplectic algebroids and their higher gauge symmetries, in terms of classical geometric “higher structures”, such as Courant algebroids twisted by Ω^2 -gerbes. As applications, we produce new examples of twisted Courant algebroids from codimension-two cycles, and we give symplectic interpretations for several well known features of higher structures (such as twists, Pontryagin classes, and tensor products). The proofs are valid in the C^∞ , holomorphic and algebraic settings, and are based on a number of technical results on the homotopy theory of L_∞ algebroids and their differential forms, which may be of independent interest.

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1 Introduction

The theory of shifted symplectic structures, introduced by Pantev, Toën, Vaquié and Vezzosi (PTVV) in [54], extends classical symplectic geometry on manifolds to the general context of derived stacks. It gives a powerful systematic framework encompassing many examples from topological field theory, algebraic integrable systems, quantum groups, enumerative geometry and more.

One challenge of the theory is that in order to have an invariant notion of symplectic structures on stacks, one must deal systematically with complexes (particularly the tangent complex), and one must take care to ensure that all constructions are invariant under quasi-isomorphisms. Thus, for instance, the strict equation $d\omega = 0$ for a closed differential form must be weakened by an infinite sequence of coherent higher homotopies. Similarly, the notion of nondegeneracy of a two-form is weakened, allowing for a quasi-isomorphism between the tangent complex \mathcal{T} and the shifted cotangent complex $\mathcal{T}^\vee[q]$, rather than a strict isomorphism. Finally, unlike their classical counterparts, shifted symplectic structures have natural internal gauge symmetries given by homotopies between forms, homotopies between homotopies, etc., leading to a whole ∞ -groupoid of symplectic structures, rather than just a set.

As a result, one desires “strictification” results that compress all of this higher homotopical data into simpler, finitary normal forms. In the case of negative shifts $q < 0$, such normal forms have been described in [8, 10, 39], with remarkable applications to categorified Donaldson–Thomas theory. In the present paper, we establish normal forms in the case of positive shifts $q \leq 2$, with a view towards classical topological field theory.

More precisely, suppose \mathcal{W} is a shifted symplectic stack (or a Lagrangian therein) and $f : X \rightarrow \mathcal{W}$ is a map from a smooth manifold or variety. We seek a normal form for the formal neighbourhood $\widehat{\mathcal{W}}$ of $f(X)$, analogous to Weinstein’s neighbourhood theorems in classical symplectic geometry [66, 67]. To establish such normal forms, we exploit the recently established equivalence between formal moduli problems and L_∞ algebroids [18, 30, 50, 53, 55]. This equivalence implies that $\widehat{\mathcal{W}}$ is isomorphic to the quotient stack $[X/\mathcal{T}_f]$, where \mathcal{T}_f is the relative tangent complex of f . Here \mathcal{T}_f is viewed as an L_∞ algebroid that acts infinitesimally on X via its natural anchor map $\mathcal{T}_f \rightarrow \mathcal{T}_X$. Conversely, given an L_∞ algebroid \mathcal{L} on X , there is a universal formal neighbourhood of X having \mathcal{L} as its relative tangent complex, namely the quotient $X \rightarrow [X/\mathcal{L}]$.

Thus, the problem at hand is to give a strict model for the full ∞ -groupoid that classifies L_∞ algebroids with shifted symplectic or Lagrangian structures. We focus in this paper on the “underived” situation, in which the stack \mathcal{W} and the map f are smooth; these correspond to L_∞ algebroids that are concentrated in nonpositive cohomological degrees.

Recent results of Calaque–Grivaux [18] and Nuiten [52, 53] give a useful algebraic model for the geometry of $[X/\mathcal{L}]$; we review some of this theory in Section 2. For instance, the Čech cohomology of the structure sheaf $\mathcal{O}_{[X/\mathcal{L}]}$ is computed by the well known Chevalley–Eilenberg algebra of \mathcal{L} , and complexes

Table 1: Classification of NQ manifolds with degree- n symplectic forms

q	NQ manifold	Differential geometric data	AKSZ field theory
0	X	X is a symplectic manifold	Classical mechanics
1	$\mathcal{T}_X^\vee[1]$	X is a Poisson manifold	Poisson sigma model
2	$M \subset \mathcal{T}_X^\vee[2]\mathcal{E}[1]$	\mathcal{E} is a Courant algebroid [48]	Courant sigma model

of quasi-coherent sheaves on $[X/\mathcal{L}]$ correspond to certain dg modules over this dg algebra (the representations up to homotopy of [1]). In particular, the tangent complex $\mathcal{T}_{[X/\mathcal{L}]}$ corresponds to the module of graded derivations, and the differential forms yield the Weil algebra of [6, 51]. (These results are closely related to the van Est maps [6, 22] for Lie groupoids.)

The study of symplectic structures in the Weil algebra of an L_∞ algebroid has a long history in the literature on differential graded (NQ) manifolds. The subject arose, in part, from topological field theory: an NQ manifold with a strictly closed and strictly nondegenerate symplectic structure of degree q can be viewed as the target space of a classical $(q+1)$ -dimensional topological field theory via the AKSZ transgression procedure [4]. Beginning with the works of Roytenberg [56] and Ševera [58], it was understood that such spaces have natural interpretations in terms of differential-geometric “higher structures”, defined by vector bundles equipped with various tensors and brackets. They gave a complete geometric classification of strict symplectic structures of degree $q \leq 2$ by relating them to degree-shifted cotangent bundles; see Table 1. The case $q = 3$ was also worked out recently by Liu and Sheng [47].

It was soon realized that low-dimensional AKSZ field theories admit various modifications and twists that do not come from strict symplectic structures. For example, Klimčík and Strobl [42] showed that one can add an extra term to the action of the Poisson sigma model using a closed three-form $H \in \Omega^3(X)$. In this way, one arrives at the notion of a twisted Poisson manifold, where the Jacobi identity for the Poisson bracket fails by a term involving H . Similarly, Hansen and Strobl [36] showed that Courant algebroids and their sigma models can be twisted by a closed four-form $K \in \Omega^4(X)$. These were given “homotopy symplectic” interpretations in [38, 44]. In a different direction, Kotov, Schaller and Strobl [43] described a model in which the nondegeneracy condition is relaxed, replacing twisted Poisson manifolds with Dirac manifolds.

In this paper, we develop several new tools for strictifying shifted symplectic algebroids, and use them to establish the first complete classification for shifts up to two, in terms similar to the above:

Theorem. Let X be a C^∞ manifold, a complex manifold or a smooth algebraic variety over a field of characteristic zero. Then q -shifted symplectic L_∞ algebroids on X for $q \leq 2$ are determined up to equivalence by the classical geometric data shown in Table 2.

Table 2: Classification of shifted symplectic L_∞ algebroids

Shift q	Structure of $X \rightarrow [X/\mathcal{L}]$		
	Unconstrained	Isotropic	Lagrangian
0	Regular foliation $\mathcal{L} \subset \mathcal{T}_X$ with symplectic leaf space	$\mathcal{L} = \mathcal{T}_X$ with no further data	$\dim X = 0$ and $\mathcal{L} = 0$
1	\mathcal{L} is a Dirac structure in an exact Courant algebroid	\mathcal{L} is a Dirac structure in $\mathcal{T}_X \oplus \mathcal{T}_X^\vee$	$\mathcal{L} \cong \mathcal{T}_X^\vee$ is the Lie algebroid of a Poisson structure
2	$\mathcal{L} \cong (\mathcal{T}_X^\vee \rightarrow \mathcal{E})$ for a twisted Courant algebroid \mathcal{E}	$\mathcal{L} \cong (\mathcal{T}_X^\vee \rightarrow \mathcal{E})$ for a Courant algebroid \mathcal{E} with no twist	$\mathcal{L} \cong \mathcal{T}_X^\vee[1]$ with the standard symplectic form and the trivial L_∞ structure

It follows that the field theories above are all instances of PTVV’s far-reaching generalization of AKSZ transgression [54].

Let us remark briefly on the classification. Notice that the zero-shifted case behaves exactly as one would expect when the quotient space $[X/\mathcal{L}]$ is a classical symplectic manifold. Notice also that the strict symplectic NQ manifolds of Table 1 appear in different columns of Table 2, depending on the shift; there is no obvious uniform way to describe the strict notions in the homotopy invariant language of shifted symplectic geometry. Indeed, at the outset we had expected to find a new higher structure at shift two that would correspond to a two-form in the Weil algebra that was only weakly nondegenerate. It would be similar to a (twisted) Courant algebroid \mathcal{E} , but with the nondegeneracy of its bilinear pairing weakened, and the map $\mathcal{T}_X^\vee \rightarrow \mathcal{E}$ replaced by a correspondence. However, as a step in the classification, we actually prove that a two-shifted symplectic algebroid can always be made strictly nondegenerate by passing to a quasi-isomorphic model—a statement that fails for other shifts.

The proof of the classification involves a collection of new technical results on the homotopy theory of L_∞ -algebroids and their differential forms that substantially simplify the practical calculations:

- Theorem 2.7 gives an efficient simplicial model for the mapping spaces in the ∞ -category of L_∞ algebroids [52, 65], using Getzler’s results [32] on gauge fixing for Maurer–Cartan sets.
- Theorem 2.8 extends the Homotopy Transfer Theorem, to show that L_∞ algebroid structures may be transferred along quasi-isomorphisms of complexes of vector bundles. This allows us to simplify shifted symplectic algebroids by replacing them with quasi-isomorphic models.

- [Theorem 3.13](#) shows that the complex $\Omega^{\geq p, \bullet}([X/\mathcal{L}])$ of homotopy closed p -forms retracts onto a much smaller complex that is amenable to computation. It implies that the whole sequence of homotopy closure data for a p -form of degree q on $[X/\mathcal{L}]$ reduces to the cocycle for $H^q(X, \Omega_X^{\geq p})$, obtained by pullback along the projection $X \rightarrow [X/\mathcal{L}]$. It explains why twists and gauge symmetries of higher structures are always described by expressions such as $\iota_x \iota_y H$, in which vectors are contracted into forms.

In fact, these strictification results allow us to model the full ∞ -groupoids of shifted symplectic algebroids, not just the objects. This is essential in two respects. Firstly, it allows us to streamline the classification using the intriguing recursive nature of shifted symplectic structures: for instance, the classification of q -shifted Lagrangian morphisms implies the classification of $(q-1)$ -shifted symplectic structures, since the latter are equivalent to Lagrangians maps to a point. Thus we can treat the two-shifted classification first, and easily work our way down.

Secondly, in the holomorphic and algebraic settings, the classification of objects and morphisms are inextricably linked: if the manifold X is not affine, then the higher gauge symmetries can appear as higher transition functions that complicate the process of gluing (descent) over an open cover. In the two-shifted case, we discover a new phenomenon, namely that the correct definition of a twisted Courant algebroid must allow the transition functions of the underlying vector bundle to be twisted by an Ω^2 -gerbe. Thus the full statement in this case is as follows:

Theorem (see [Section 5.1](#) and [Theorem 5.5](#)). The ∞ -groupoid of two-shifted symplectic L_∞ algebroids on X is equivalent to the following strict two-groupoid:

Objects are Courant algebroids \mathcal{E} twisted by classes in $H^2(X, \Omega_X^{\geq 2})$.

1-morphisms are given by maps $f: \mathcal{E}_1 \rightarrow \mathcal{E}_2$ that preserve the pairing and anchors identically, and preserve the brackets and gluing maps up to a coboundary $H \in C^1(X, \Omega_X^{\geq 2})$.

2-morphisms are elements $B \in C^0(X, \Omega_X^2)$ that shear the bundle maps

In [Section 5.4](#), we use the theorem to give new examples of twisted Courant algebroids. In particular, in the holomorphic/algebraic setting, we show that any smooth codimension-two cycle $Y \subset X$ gives rise to a twisted Courant algebroid whose twisting class is the cycle class $[Y] \in H^{2,2}(X)$; the procedure is reminiscent of Serre’s construction of rank-two vector bundles. We expect that this construction will allow fivebranes (curves in Calabi–Yau threefolds) to be incorporated into the recent works [\[5, 7, 26, 31\]](#) relating holomorphic Courant algebroids to heterotic string theory.

[Theorem 6.3](#) gives an analogous description of the ∞ -groupoid of two-shifted Lagrangians $[Y/\mathcal{M}] \rightarrow [X/\mathcal{L}]$ where $Y \subset X$ is a closed submanifold. Now the objects are pairs $(\mathcal{E}, \mathcal{F})$ of a twisted Courant algebroid \mathcal{E} on X and a maximal isotropic twisted subbundle $\mathcal{F} \subset \mathcal{E}|_Y$ that is involutive for the bracket. This

extends the notion of a Dirac structure with support [3, 16, 58], allowing for twists by relative cohomology classes of the pair (X, Y) .

From this we immediately deduce the classification of one-shifted symplectic structures in terms of Dirac structures in exact Courant algebroids (Theorem 7.1). This approach illuminates the dual nature of such Dirac structures: on the one hand, they correspond to Lagrangians in certain degree-two symplectic NQ manifolds [57]; on the other hand, they are the linearizations of quasi-symplectic Lie groupoids [15, 69], and by definition, the latter are exactly the atlases for one-shifted symplectic stacks.

Throughout the paper we explain how numerous well known features of Dirac structures and Courant algebroids can be viewed through the unifying lens of shifted symplectic geometry. For instance, the classification of isotropic quotients immediately recovers Ševera’s cohomological classification [57] of exact Courant algebroids (Corollary 5.10), as well as the link [11, 57] between Courant extensions and the first Pontryagin class (Corollary 5.11). The Courant trivializations and “generalized tangent bundles” defined in the context of generalized complex branes [34] are interpreted as Lagrangians (Corollary 6.6 and Section 7.2). Finally, the formalism of derived Lagrangian intersections gives a perspective on both the tensor product [2, 34] of Dirac structures (Section 7.1), and the equivalence [24, 68] between symplectic groupoids and Poisson manifolds (Section 7.3).

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2 Basics of L_∞ algebroids

Throughout the paper, a *manifold* is a real C^∞ manifold, a complex manifold, or a smooth algebraic variety over a field of characteristic zero. We denote by \mathbb{K} the base field, which is \mathbb{R} , \mathbb{C} , or an arbitrary field \mathbb{K} of characteristic zero, respectively. The structure sheaf \mathcal{O}_X is the corresponding sheaf of C^∞ , holomorphic, or algebraic functions.

We also consider the subclass of *affine manifolds*, which are arbitrary C^∞ -manifolds, Stein manifolds, and smooth affine varieties, respectively. The essential features of affine manifolds that we need are that extensions of vector bundles always split, and appropriate sheaves of modules are acyclic for sheaf cohomology. Moreover, every manifold has a good cover by affine manifolds. By a quasi-coherent sheaf on an affine manifold we will mean a module over the algebra of global functions.

We adopt the convention that a vector bundle is the same as a locally-free sheaf, i.e. we allow the possibility that a bundle may have different rank

on different connected components. With this definition understood, vector bundles on an affine manifold are equivalent to finite-rank projective modules over the algebra of global functions.

In this section, we cover the basic definitions and properties of the category of L_∞ algebroids. We focus first on the affine case, and then extend to general manifolds by gluing along affine covers.

2.1 Affine L_∞ algebroids

Let U be an affine manifold, and consider a bounded complex

$$\mathcal{L} = \left(\cdots \xrightarrow{\delta} \mathcal{L}_2 \xrightarrow{\delta} \mathcal{L}_1 \xrightarrow{\delta} \mathcal{L}_0 \right)$$

of finite rank vector bundles, where \mathcal{L}_i sits in degree $-i$. Here and throughout, we make no notational distinction between a vector bundle and its corresponding finite rank projective $\mathcal{O}(U)$ -module of sections.

Definition 2.1. An L_∞ **algebroid structure on \mathcal{L}** is an L_∞ algebra structure on the \mathbb{K} -vector space \mathcal{L} , determined by a \mathbb{K} -bilinear bracket

$$[-, -] : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$$

of degree zero and a collection of $\mathcal{O}(U)$ -multilinear brackets

$$[-, \dots, -] : \underbrace{\mathcal{L} \times \cdots \times \mathcal{L}}_{n \text{ times}} \rightarrow \mathcal{L}, \quad n > 2$$

of degree $2 - n$, together with an $\mathcal{O}(U)$ -linear morphism of complexes

$$a : \mathcal{L} \rightarrow \mathcal{T}_U,$$

subject to the Leibniz rule

$$[x, fy] = (\mathcal{L}_a x)f y + f[x, y]$$

for $x, y \in \mathcal{L}$ and $f \in \mathcal{O}(U)$. An L_∞ algebroid is a **Lie n -algebroid on U** if the underlying complex is concentrated in degrees $[-(n-1), \dots, 0]$.

Remark 2.2. We remark that this notion of an L_∞ algebroid is often referred to as a *split* L_∞ algebroid; see, e.g., [61]. \square

Remark 2.3. There is a more general notion of L_∞ algebroid, in which the anchor map is supplemented with a sequence of higher homotopies, making it into an L_∞ morphism. Such homotopies do not occur in our situation for degree reasons, but they would occur if we allowed the complex \mathcal{L} is to have terms in positive cohomological degrees, or replaced the manifold U with something more singular (e.g. a derived scheme). See, for instance [17, 40, 71] where this viewpoint is used to model formal neighbourhoods of subvarieties. \square

In what follows, we will often suppress the anchor and brackets from notation and simply say that \mathcal{L} is an L_∞ algebroid on U . We will also occasionally write $[-]_1 = \delta$ for the differential on the underlying complex.

We recall that by an L_∞ structure, we mean that the brackets are graded skew-symmetric, and that the higher Jacobi identity

$$\sum_{i+j=n+1} \sum_{\sigma \in S(i,j-1)} (-1)^{\sigma+\epsilon} (-1)^{i(j-1)} [[x_{\sigma_1}, \dots, x_{\sigma_i}], x_{\sigma_{i+1}}, \dots, x_{\sigma_n}] = 0$$

holds for all n . Here $S(i, j-1)$ denotes the set of $(i, j-1)$ -unshuffles, $(-1)^\sigma$ is the sign of the permutation σ , and $(-1)^\epsilon$ is determined by the Koszul sign rule.

Given an L_∞ algebroid \mathcal{L} on U we define a commutative differential graded algebra (cdga) known as the **Chevalley–Eilenberg algebra** to be

$$C^\bullet(\mathcal{L}) := (\text{Sym}_{\mathcal{O}(U)}(\mathcal{L}^\vee[-1]), \delta).$$

Here $\mathcal{L}^\vee[-1]$ is the dual complex, with a shift, and Sym denotes the graded symmetric algebra. When \mathcal{L} is concentrated in degree zero, we have that $C^\bullet(\mathcal{L}) = \wedge^\bullet \mathcal{L}^\vee$ with \mathcal{L}^\vee in degree one. But in general, $C^\bullet(\mathcal{L})$ has two distinct gradings: the first is the internal degree induced by the grading on \mathcal{L} , and the second is the *weight*, defined so that the n th symmetric power has weight n . Thus an element of \mathcal{L}_j^\vee defines a generator for $C^\bullet(\mathcal{L})$ of degree $j+1$ and weight one. We denote the weight- n component of an element $u \in C^\bullet(\mathcal{L})$ by u_n .

The differential δ on $C^\bullet(\mathcal{L})$ has degree one, but components of many weights, corresponding to the differential on \mathcal{L} and the higher Lie brackets. Viewing elements $u \in C^\bullet(\mathcal{L})$ as multilinear operators on \mathcal{L} , the differential δu is defined by its weight components

$$(\delta u)_n(x_1, \dots, x_n) = (\delta_{\text{CE}} u)_n(x_1, \dots, x_n) + \sum_{i=1}^n (-1)^\epsilon \mathcal{L}_{ax_i} u_{n-1}(x_1, \dots, \widehat{x_i}, \dots, x_n)$$

where $\delta_{\text{CE}} u$ is the \mathbb{K} -multilinear operator $\mathcal{L} \times \dots \times \mathcal{L} \rightarrow \mathcal{O}(U)$ defined by summing over all possible insertions of brackets into u :

$$(\delta_{\text{CE}} u)_n(x_1, \dots, x_n) = \sum_{i+j=n+1} \sum_{\sigma \in S(i,j-1)} (-1)^{\binom{i}{2}+\epsilon} u_j([x_{\sigma_1}, \dots, x_{\sigma_i}]_i, x_{\sigma_{i+1}}, \dots, x_{\sigma_n}). \quad (1)$$

Thus δ is simply the Chevalley–Eilenberg differential of $\mathcal{O}(U)$ when viewed as a module over the L_∞ algebra \mathcal{L} , but restricted to the $\mathcal{O}(U)$ -multilinear cochains. By [9], this formula gives a bijective correspondence between L_∞ algebroid structures on \mathcal{L} and differentials on $\text{Sym}(\mathcal{L}^\vee[-1])$ that make it into a cdga (often called Q -structures or homological vector fields). We refer to [45] for an example of L_∞ algebroids arising from singular foliations.

2.2 Morphisms of affine L_∞ algebroids

In this section, we explain how the natural notions of morphisms between L_∞ algebroids, homotopies between morphisms, homotopies between homotopies,

etc., lead to the construction of a simplicial category whose objects are L_∞ algebroids over a given affine manifold U . We remark that for many purposes, it is better to take an alternative route to constructing the ∞ -category of L_∞ algebroids, realizing it as the ∞ -categorical localization of a suitable model category of L_∞ algebroids, in which the weak equivalences are quasi-isomorphisms. This alternative approach was developed in [52] and [65], and it necessitates working with a larger class of L_∞ algebroids than the ones we use in this paper. However, by [52, Corollary 5.14], the resulting mapping spaces are ultimately equivalent to the ones we describe below. Hence the main result of this section can be understood as giving an efficient “strict” model for the homotopically correct mapping spaces. See also [27] for the simplicial category of L_∞ algebras.

We begin by recalling the natural notion of morphisms:

Definition 2.4. Let \mathcal{L} and \mathcal{M} be L_∞ algebroids over an affine manifold U . A *(base-preserving) L_∞ morphism* $f : \mathcal{M} \rightarrow \mathcal{L}$ is a morphism of cdgas

$$f^* : C^\bullet(\mathcal{L}) \rightarrow C^\bullet(\mathcal{M})$$

that acts as the identity on the weight-zero part, i.e. on $\mathcal{O}(U)$.

By [9], such a morphism may be described explicitly by a sequence of $\mathcal{O}(U)$ -multilinear graded-skew-symmetric maps

$$f_n : \underbrace{\mathcal{M} \times \cdots \times \mathcal{M}}_{n \text{ times}} \rightarrow \mathcal{L}, \quad n \geq 1$$

of degree $1 - n$, with the following properties:

1. $f_1 : \mathcal{M} \rightarrow \mathcal{L}$ is a morphism of complexes of vector bundles that is compatible with the anchors, i.e. $a_{\mathcal{M}} f_1 = a_{\mathcal{L}}$.
2. The sequence f_1, f_2, \dots , defines an L_∞ morphism of the \mathbb{K} -linear L_∞ algebras underlying \mathcal{L} and \mathcal{M} , i.e. we have the usual bracket compatibilities

$$\begin{aligned} & \sum_{i+j=n+1} \sum_{\sigma \in S(i, j-1)} (-1)^{\sigma+(i-1)j} f_j([x_{\sigma_1}, \dots, x_{\sigma_i}], x_{\sigma_{j+1}}, \dots, x_{\sigma_n}) \\ &= \sum_{\substack{j \geq 1 \\ i_1 + \dots + i_j = k}} \sum_{\sigma \in S(i_1, \dots, i_k)} (-1)^{\sigma+\tau} [f_{i_1}(x_1, \dots, x_{i_1}), \dots, f_{i_j}(x_{\sigma_{n-i_j+1}}, \dots, x_{\sigma_n})] \end{aligned}$$

where $\tau = (j-1)(i_1-1) + \dots + 1 \cdot (i_{j-1}-1)$.

Definition 2.5. An L_∞ morphism f is a *weak equivalence* if its linear component $f_1 : \mathcal{M} \rightarrow \mathcal{L}$ is a quasi-isomorphism.

Higher homotopies between morphisms are defined by the following standard construction. Let $\Omega_n = \Omega^\bullet(\Delta^n)$ be the algebra of polynomial functions on the n -simplex, i.e.

$$\Omega^n = \mathbb{K}[t_0, \dots, t_n, dt_0, \dots, dt_n] / \left(\sum t_i = 1, \sum dt_i = 0 \right)$$

The collection Ω_\bullet forms a simplicial cdga with the usual face and degeneracy maps, which allows us to define a whole simplicial set of morphisms $\text{Hom}(\mathcal{M}, \mathcal{L})_\bullet$:

Definition 2.6. The n -simplices of the simplicial set $\text{Hom}(\mathcal{M}, \mathcal{L})_\bullet$ are the Ω_n -linear cdga maps $\Omega_n \otimes \mathbf{C}^\bullet(\mathcal{L}) \rightarrow \Omega_n \otimes \mathbf{C}^\bullet(\mathcal{M})$ that act trivially on the summand $\Omega_n \otimes \mathcal{O}(U)$.

By construction an n -simplex of $\text{Hom}(\mathcal{M}, \mathcal{L})_\bullet$ is an L_∞ -morphism of Ω_n -linear L_∞ algebroids $\mathcal{M} \otimes \Omega_n \rightarrow \mathcal{L} \otimes \Omega_n$. Due to Ω_n -linearity this is the same as a morphism of \mathbb{K} -linear L_∞ algebras $\mathcal{M} \rightarrow \mathcal{L} \otimes_{\mathbb{K}} \Omega_n$ such that the constituent maps $f_i: \wedge^i \mathcal{M} \rightarrow \mathcal{L} \otimes \Omega_n$ are $\mathcal{O}(U)$ -multilinear, and the component $f_1: \mathcal{M} \rightarrow \mathcal{L} \otimes \Omega_n$ is compatible with the anchors. In other words, f_1 projects to $a_{\mathcal{M}} \otimes 1$ under the natural map

$$\text{Hom}_{\mathbb{K}}(\mathcal{M}, \mathcal{L} \otimes \Omega_n) = \text{Hom}_{\mathbb{K}}(\mathcal{M}, \mathcal{L}) \otimes \Omega_n \rightarrow \text{Hom}(\mathcal{M}, \mathcal{T}_U) \otimes \Omega_n$$

given by composing with the anchor $a_{\mathcal{L}}: \mathcal{L} \rightarrow \mathcal{T}_U$.

The simplicial set $\text{Hom}(\mathcal{M}, \mathcal{L})_\bullet$ is quite large, so it is useful to have a more efficient model. To this end, we follow Getzler and use Dupont's gauge operator

$$s_n: \Omega^k(\Delta^n) \rightarrow \Omega^{k-1}(\Delta^n);$$

to cut down on the redundancy; we refer to [32, Section 3] for the detailed description of this operator. Restricting to morphisms that satisfy the gauge condition

$$(s_\bullet \otimes \text{id})f^*(1 \otimes g) = 0$$

for any $g \in \mathbf{C}^\bullet(\mathcal{L})$, we obtain a simplicial subset

$$\text{Hom}_{\text{red}}(\mathcal{M}, \mathcal{L})_\bullet \subset \text{Hom}(\mathcal{M}, \mathcal{L})_\bullet$$

whose relevance is illustrated by the following result. Recall from [32, Definition 2.4] the notion of an n -groupoid which is a special Kan complex.

Theorem 2.7. *Let U be an affine manifold. For any L_∞ algebroids \mathcal{L} and \mathcal{M} on U , the following statements hold:*

1. *The simplicial set $\text{Hom}(\mathcal{M}, \mathcal{L})_\bullet$ is a Kan complex, i.e. a weak ∞ -groupoid.*
2. *The inclusion $\text{Hom}_{\text{red}}(\mathcal{M}, \mathcal{L})_\bullet \subset \text{Hom}(\mathcal{M}, \mathcal{L})_\bullet$ is a weak equivalence.*
3. *If \mathcal{M} is a Lie n -algebroid, then $\text{Hom}_{\text{red}}(\mathcal{M}, \mathcal{L})_\bullet$ is an $(n-1)$ -groupoid.*

Proof. If $\text{Hom}(\mathcal{M}, \mathcal{L})_\bullet$ is empty, the statements are vacuous. Thus we may assume that it is nonempty and fix an L_∞ algebroid morphism $f: \mathcal{M} \rightarrow \mathcal{L}$. Using f , we will express $\text{Hom}(\mathcal{M}, \mathcal{L})_\bullet$ as the Maurer–Cartan simplicial set $\text{MC}_\bullet(\mathfrak{h})$ of a nilpotent L_∞ algebra \mathfrak{h} concentrated in degrees $-(n-1), \infty$, so that the three statements follow directly from Proposition 4.7, Corollary 5.11 and Theorem 5.4 in [32], respectively.

To begin, consider the simplicial set $\text{Hom}_{\mathbb{K}}(\mathcal{M}, \mathcal{L})_\bullet$ of all \mathbb{K} -linear L_∞ morphisms $\mathcal{M} \rightarrow \mathcal{L} \otimes \Omega_\bullet$. We recall from, e.g. [28, Section 3] or [62, Section 3], that $\text{Hom}_{\mathbb{K}}(\mathcal{M}, \mathcal{L})_\bullet$ is the Maurer–Cartan simplicial set of a \mathbb{K} -linear L_∞ algebra

$$\mathfrak{g} = \text{Hom}_{\mathbb{K}}(\text{Sym}_{\mathbb{K}}^{\geq 1}(\mathcal{M}[1]), \mathcal{L}).$$

The action of the differential $\delta_{\mathfrak{g}}$ on $\psi \in \mathfrak{g}$ is given by its weight components

$$(\delta_{\mathfrak{g}}\psi)(x_1, \dots, x_m) = \delta_{\mathcal{L}}(\psi_m(x_1, \dots, x_m)) + (\delta_{\text{CE}}\psi)(x_1, \dots, x_m) \quad (2)$$

where δ_{CE} is defined by the same formula as in (1). Meanwhile, the binary bracket of $\psi, \phi \in \mathfrak{g}$ is given by

$$[\psi, \phi]_{\mathfrak{g}}(x_1, \dots, x_m) = \sum_{i+j=m} \sum_{\sigma \in S(i,j)} (-1)^{\epsilon} [\psi_i(x_{\sigma_1}, \dots, x_{\sigma_i}), \phi_j(x_{\sigma_{i+1}}, \dots, x_{\sigma_m})]_{\mathcal{L}}. \quad (3)$$

where $x_1, \dots, x_m \in \mathcal{M}$, and the higher brackets are defined similarly, by composition with the higher brackets on \mathcal{L} .

Consider the vector subspace

$$\mathfrak{h} = \text{Hom}_{\mathcal{O}(U)}(\text{Sym}_{\mathcal{O}(U)}^{\geq 1}(\mathcal{M}[1]), \mathcal{K}) \subset \mathfrak{g}$$

consisting of $\mathcal{O}(U)$ -multilinear maps that take values in $\mathcal{K} = \ker a_{\mathcal{L}} \subset \mathcal{L}$. Evidently elements of $\text{Hom}(\mathcal{M}, \mathcal{L})_{\bullet}$ can be identified with Maurer–Cartan elements in $\mathfrak{g} \otimes \Omega_{\bullet}$ of the form $f \otimes 1 + g$ where $g \in \mathfrak{h} \otimes \Omega_{\bullet}$. Since f itself is a Maurer–Cartan element, g must be a Maurer–Cartan element for the f -twisted L_{∞} brackets

$$[\psi_1, \dots, \psi_k]_{\mathfrak{g}, f} = \sum_{l \geq 0} \frac{1}{l!} \underbrace{[f, \dots, f]_{l \text{ times}}}_{l \text{ times}}, \psi_1, \dots, \psi_k]_{\mathfrak{g}}. \quad (4)$$

Thus to identify $\text{Hom}(\mathcal{M}, \mathcal{L})_{\bullet} = \text{MC}_{\bullet}(\mathfrak{h})$, it is sufficient to show that $\mathfrak{h} \subset \mathfrak{g}$ is preserved by the twisted brackets.

To this end, suppose that $\psi_1, \dots, \psi_k \in \mathfrak{h}$ and consider the \mathbb{K} -linear map

$$[\psi_1, \dots, \psi_k]_{\mathfrak{g}, f}: \text{Sym}_{\mathbb{K}}^{\geq 1} \mathcal{M}[1] \rightarrow \mathcal{L}.$$

Applying the anchor of \mathcal{L} to (2) and (3), we see that this linear map automatically takes values in $\mathcal{K} \subset \mathcal{L}$, so it remains to check that it is $\mathcal{O}(U)$ -multilinear. For this, we observe that if $k \geq 2$, then every term on the right hand side of (4) is an $\mathcal{O}(U)$ -multilinear operator; indeed, such a term is either a binary bracket $[\psi_1, \psi_2]$, which is $\mathcal{O}(U)$ -bilinear because ψ_1, ψ_2 are valued in \mathcal{K} , or it is a higher bracket, and therefore automatically multilinear. Similarly, for $k = 1$, we have the twisted differential

$$\begin{aligned} (\delta_{\mathfrak{g}, f}\psi)(x_1, \dots, x_m) &= - \sum_{i < j} (-1)^{\epsilon} \psi_m([x_i, x_j]_{\mathcal{M}}, x_1, \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots, x_m) \\ &\quad + \sum_i (-1)^{\epsilon} [f(x_i), \psi_m(x_1, \dots, \widehat{x}_i, \dots, x_m)]_{\mathcal{L}} + \dots \end{aligned}$$

where \dots denotes terms that are manifestly $\mathcal{O}(U)$ -multilinear. Using the Leibniz rule for the binary brackets on \mathcal{L} and \mathcal{M} and the fact that f intertwines the anchors, we see that the nonlinearities cancel, as desired. \square

This construction can be easily generalized to maps that do not preserve the base. Suppose that $f: V \rightarrow U$ is a morphism of affine manifolds, and \mathcal{L} and \mathcal{M} are L_∞ algebroids on U and V . A morphism $\mathcal{M} \rightarrow \mathcal{L}$ covering f is an extension of $f^*: \mathcal{O}(U) \rightarrow \mathcal{O}(V)$ to a morphism of cdgas $C^\bullet(\mathcal{L}) \rightarrow C^\bullet(\mathcal{M})$, with higher homotopies defined by tensoring with forms on simplices as above. We refer to [9, Definition 4.1.6] for a more explicit description of morphisms in terms of brackets. The conclusion is that every morphism $\mathcal{M} \rightarrow \mathcal{L}$ covering f factors uniquely through the pullback L_∞ algebroid $f^!\mathcal{L}$, defined on the level of complexes by the fibre product

$$f^!\mathcal{L} = \mathcal{L} \times_{f^*\mathcal{T}_U} \mathcal{T}_V.$$

Hence this more general situation reduces to the base-preserving one.

2.3 Homotopy transfer

Recall the Homotopy Transfer Theorem: if two complexes of vector spaces are quasi-isomorphic, and one of them is an L_∞ algebra, then so is the other, and the quasi-isomorphism extends to an equivalence of the L_∞ -structures. See, for example, [49, Theorem 10.3.9]. In this section extend this result to algebroids:

Theorem 2.8. *Let \mathcal{L}' and \mathcal{L}'' be bounded complexes of vector bundles on an affine manifold U , and let $f: \mathcal{L}' \rightarrow \mathcal{L}''$ be a quasi-isomorphism. Given an L_∞ algebroid structure on \mathcal{L}'' , there exists an L_∞ algebroid structure on \mathcal{L}' and an extension of f to an L_∞ quasi-isomorphism.*

Recall that a **special deformation retract** between bounded complexes \mathcal{L} and \mathcal{M} is a pair of morphisms

$$\mathcal{L} \begin{array}{c} \xrightarrow{i} \\ \xleftarrow{p} \end{array} \mathcal{M}$$

and a homotopy operator $h: \mathcal{M} \rightarrow \mathcal{M}[-1]$ satisfying the identities

$$ip - \text{id} = \delta h + h\delta, \quad pi = \text{id}, \quad hi = 0, \quad ph = 0, \quad h^2 = 0.$$

An arbitrary quasi-isomorphism of complexes may be factored into a pair of special deformation retracts using the mapping cylinder construction. Thus, it is enough to prove Theorem 2.8 in the case where the map $f: \mathcal{L}' \rightarrow \mathcal{L}''$ is one of the maps $i: \mathcal{L} \rightarrow \mathcal{M}$ or $p: \mathcal{M} \rightarrow \mathcal{L}$ in a special deformation retract. These two cases are the content of the following two lemmas.

Lemma 2.9. *If \mathcal{L} is an L_∞ algebroid, then there exists an L_∞ algebroid structure on \mathcal{M} such that both $i: \mathcal{L} \rightarrow \mathcal{M}$ and $p: \mathcal{M} \rightarrow \mathcal{L}$ are L_∞ quasi-isomorphisms.*

Proof. Use the deformation retract to split

$$\mathcal{M} \cong \mathcal{L} \oplus \mathcal{K}$$

where $\mathcal{K} = \ker p$ is a contractible complex. Therefore, we can identify

$$\mathcal{K} \cong \text{Cone}(\mathcal{K}' \xrightarrow{\text{id}} \mathcal{K}')$$

for a complex of vector bundles \mathcal{K}' .

Pick a (not necessarily flat) \mathcal{L} -connection on \mathcal{K}' . Then by [1, Example 3.8] we obtain the data of a representation up to homotopy on \mathcal{K} , i.e. a differential on $\text{Sym}(\mathcal{L}^\vee[-1]) \otimes \mathcal{K}$ making it into a dg module over the dg algebra $\mathcal{O}([U/\mathcal{L}])$. Therefore we may extend the differential on $\text{Sym}(\mathcal{L}^\vee[-1]) \otimes \mathcal{K}$ to the symmetric algebra

$$\text{Sym}(\mathcal{M}^\vee[-1]) \cong \text{Sym}(\mathcal{L}^\vee[-1]) \otimes_{\mathcal{O}(U)} \text{Sym}(\mathcal{K}^\vee[-1])$$

resulting in an L_∞ algebroid structure on \mathcal{M} . By construction, the projection

$$\text{Sym}(i^*): \text{Sym}(\mathcal{M}^\vee[-1]) \rightarrow \text{Sym}(\mathcal{L}^\vee[-1])$$

is compatible with the differentials and similarly for $\text{Sym}(p^*)$. Hence p and i define strict quasi-isomorphisms of L_∞ algebroids. \square

Lemma 2.10. *If \mathcal{M} is an L_∞ algebroid, then there exists an L_∞ algebroid structure on \mathcal{L} and an extension of i to an L_∞ quasi-isomorphism.*

Proof. We define an anchor on \mathcal{L} by the composite ai where a is the anchor on \mathcal{M} . To define the brackets, we use the homotopy transfer theorem for L_∞ algebras (e.g. [49, Theorem 10.3.9]). It allows us to transfer the \mathbb{K} -linear L_∞ algebra structure on \mathcal{M} to one on \mathcal{L} , and to extend i to a sequence of maps

$$i_n: \underbrace{\mathcal{L} \times \cdots \times \mathcal{L}}_{n \text{ times}} \rightarrow \mathcal{M}$$

giving a \mathbb{K} -linear L_∞ -quasi-isomorphism.

By inspection of the explicit formulae for the transferred structure, we see that the maps i_n are $\mathcal{O}(U)$ -multilinear due to the equations $h^2 = 0$ and $hi = 0$. The binary bracket is

$$[x, y]_{\mathcal{L}} = p[ix, iy]_{\mathcal{M}}$$

and since $pi = \text{id}$ we see that it satisfies the correct Leibniz rule. Finally, the higher brackets on \mathcal{L} are manifestly \mathcal{O} -linear since those of \mathcal{M} are. \square

2.4 Globalization

We now give a definition of L_∞ algebroids valid for arbitrary manifolds X . The idea is to glue together L_∞ algebroids defined on affine subsets as above. Crucially, we are allowed to glue by arbitrary quasi-isomorphisms, rather than just strict isomorphisms. This presents the difficulty that the cocycle condition may be satisfied only up to higher homotopies which in turn satisfy an infinite sequence of coherences, which may be succinctly summarized as follows.

By the results of Section 2.2 or the model categorical approach of [52, 65], L_∞ algebroids on an affine manifold U form an ∞ -category $L_\infty \text{Alg}(U)$ for which

the mapping spaces are modelled by the simplicial sets $\mathrm{Hom}(\mathcal{L}, \mathcal{M})_\bullet$. Given an open embedding of affine manifolds $i: U \rightarrow V$, there is an obvious pullback functor

$$i^*: L_\infty \mathrm{Algd}(V) \rightarrow L_\infty \mathrm{Algd}(U)$$

given by restricting bundles, anchors, brackets, etc.

Definition 2.11. Let X be a manifold. The ∞ -category of L_∞ algebroids on X is the homotopy limit

$$L_\infty \mathrm{Algd}(X) = \lim_{U \subset X} L_\infty \mathrm{Algd}(U).$$

over affine open subsets $U \subset X$.

Notice that if X is an affine manifold, the category of affine open subsets has a final object; hence this definition recovers the original definition. One can show that $L_\infty \mathrm{Algd}(-)$ satisfies descent, i.e. it is an ∞ -sheaf. This allows one to describe an L_∞ algebroid on a general manifold X by choosing an affine cover $\{U_i\} \rightarrow X$, putting an L_∞ algebroid \mathcal{L}_i on each U_i , a quasi-isomorphism $g_{ij}: \mathcal{L}_i \rightarrow \mathcal{L}_j$ on each double overlap $U_i \cap U_j$, a homotopy $h_{ijk}: g_{ij}g_{jk}g_{ki} \Rightarrow 1$ on each triple overlap $U_i \cap U_j \cap U_k$, etc., satisfying an appropriate cocycle condition. We will not use this statement in the paper

Remark 2.12. If X is a complex manifold or smooth algebraic variety that is not quasi-projective, then an L_∞ algebroid on X may not have an underlying global complex of vector bundles, essentially because coherent sheaves on X may not have global resolutions by vector bundles. However, if \mathcal{L} is a Lie 1-algebroid, it is defined by a global vector bundle, giving a Lie algebroid in the classical sense. \square

Remark 2.13. We note that a similar approach to globalization has been used in [64] in the case of sheaves L_∞ algebras, in order to construct homotopy L_∞ spaces that model the deformation theory of bundles on compact complex manifolds. Alternatively, one can give a more global definition of L_∞ algebroids using Čech/Thom–Whitney resolutions as in [17], or using Dolbeault resolutions as in [71] (in the case of complex manifolds). It follows from our abstract definition of $L_\infty \mathrm{Algd}(X)$ as a limit that such objects will give rise to L_∞ algebroids in our sense. However, since mapping spaces in those models are not discussed, we cannot make a precise statement about the resulting ∞ -categories. \square

3 Geometry of the quotient

In [18] and [53], it is shown that L_∞ algebroids are equivalent to formal moduli problems, i.e. they model formal neighbourhoods of manifolds in a higher stacks. In particular, given an L_∞ algebroid \mathcal{L} over an affine manifold U , there is a higher formal stack $[U/\mathcal{L}]$ with a projection map $\pi: U \rightarrow [U/\mathcal{L}]$ that satisfies the following property. Namely, if Y is a derived stack which admits a tangent

complex and $f: U \rightarrow Y$ is a morphism, then a factorization of the form

$$\begin{array}{ccc} U & \xrightarrow{f} & Y \\ \pi \downarrow & \nearrow & \\ [U/\mathcal{L}] & & \end{array}$$

is equivalent to a morphism of L_∞ algebroids $\mathcal{L} \rightarrow \mathcal{T}_f$, where \mathcal{T}_f is the relative tangent complex of f . For example, if \mathcal{L} is a Lie 1-algebroid, we can identify $[U/\mathcal{L}]$ with the quotient of U by the formal Lie groupoid that integrates \mathcal{L} . We also refer to [33] for an alternative approach which describes $[U/\mathcal{L}]$ as an L_∞ space.

The results of [18, 53] show that many features of the geometry of the quotient stack have algebraic descriptions in terms of the L_∞ algebroid \mathcal{L} itself. For instance, the space of global functions on $[U/\mathcal{L}]$ is the space of \mathcal{L} -invariant functions on U , which is exactly the space of zero-cocycles in the Chevalley–Eilenberg algebra $C^\bullet(\mathcal{L})$. More generally, the derived global sections of the structure sheaf $\mathcal{O}_{[U/\mathcal{L}]}$ are given by

$$\mathcal{O}([U/\mathcal{L}]) := \mathbb{R}^\bullet \Gamma([U/\mathcal{L}], \mathcal{O}_{[U/\mathcal{L}]}) \cong C^\bullet(\mathcal{L})$$

as dg \mathbb{K} -algebras (with the weight grading and the $\mathcal{O}(U)$ -module structure forgotten). We now explain how to perform calculations with objects such as the tangent complex in similar terms.

3.1 Quasi-coherent sheaves

Let U be an affine manifold and let \mathcal{L} an L_∞ algebroid over U . There is a general definition of the ∞ -category of quasi-coherent complexes on a derived stack such as $[U/\mathcal{L}]$; an object of this ∞ -category is given by specifying a dg $\mathcal{O}(V)$ -module for every map $V \rightarrow [U/\mathcal{L}]$ from an affine manifold V , in a way that is compatible with pullbacks.

It is shown in [53, Corollary 7.17] that this ∞ -category is equivalent to the ∞ -category whose objects are representations of \mathcal{L} , i.e. complexes \mathcal{E}_0 of quasi-coherent sheaves on U , equipped with an action of \mathcal{L} . This action can equivalently be described by a differential on $\mathcal{E}_0 \otimes_{\mathcal{O}(U)} C^\bullet(\mathcal{L})$, making it into a dg module over the Chevalley–Eilenberg algebra; this is the approach of “representations up to homotopy” [1]. As is well known, there are many complexes (such as the tangent complex) that are only non-canonically isomorphic to a representation up to homotopy. This motivates a slightly different, but ultimately equivalent definition, as follows.

Consider the ideal $\mathcal{I} \subset C^\bullet(\mathcal{L})$ of positive weight elements, and the induced filtration by subcomplexes

$$C^\bullet(\mathcal{L}) \supset \mathcal{I} \supset \mathcal{I}^2 \supset \dots$$

The associated graded complex is canonically isomorphic to the symmetric algebra $\text{Sym}(\mathcal{L}^\vee[-1])$, but with differential induced by the original differential δ on the complex of vector bundles \mathcal{L} , and all higher brackets forgotten.

Now let (\mathcal{E}, δ) be a filtered dg module over $C^\bullet(\mathcal{L})$, i.e. it has a filtration

$$\mathcal{E}^0 = \mathcal{E} \supset \mathcal{E}^1 \supset \mathcal{E}^2 \supset \dots,$$

compatible with the weight filtration on $C^\bullet(\mathcal{L})$. We say that \mathcal{E} is **quasi-coherent complex** if the natural map

$$(\mathcal{E}^0/\mathcal{E}^1) \otimes_{\mathcal{O}(U)} \mathcal{I}^n/\mathcal{I}^{n+1} \rightarrow \mathcal{E}^n/\mathcal{E}^{n+1}$$

is a quasi-isomorphism for all n . The point of this definition is that a quasi-coherent complex \mathcal{E} becomes a representation up to homotopy once we make a non-canonical choice of splitting of the filtration, which gives the following:

Proposition 3.1. *There is a canonical equivalence between $\text{QCoh}([U/\mathcal{L}])$ and the ∞ -category of quasi-coherent filtered dg $C^\bullet(\mathcal{L})$ -modules.*

In this paper, we will make no notational distinction between complexes on $[U/\mathcal{L}]$ and their corresponding filtered dg modules. We remark that the pullback functor $\pi^* : \text{QCoh}([U/\mathcal{L}]) \rightarrow \text{QCoh}(U)$ is given at the level of dg modules by

$$\pi^*\mathcal{E} = \mathcal{E} \otimes_{C^\bullet(\mathcal{L})} \mathcal{O}(U) \cong \mathcal{E}^0/\mathcal{E}^1, \quad (5)$$

which has the following immediate (but useful) consequence:

Lemma 3.2. *Let \mathcal{L} be an L_∞ algebroid on U , and suppose that $\phi : \mathcal{E} \rightarrow \mathcal{E}'$ is a morphism in $\text{QCoh}([U/\mathcal{L}])$. Then ϕ is an equivalence if and only if the pullback $\pi^*\phi : \pi^*\mathcal{E} \rightarrow \pi^*\mathcal{E}'$ is a quasi-isomorphism of complexes of $\mathcal{O}(U)$ -modules. In other words, π^* is a conservative ∞ -functor.*

If \mathcal{L} is an L_∞ algebroid on an arbitrary manifold X , we have

$$\text{QCoh}([X/\mathcal{L}]) = \lim_{U \subset X} \text{QCoh}([U/\mathcal{L}|_U])$$

where the limit is taken over all affine open submanifolds. Thus an object $\mathcal{E} \in \text{QCoh}([X/\mathcal{L}])$ corresponds, on each affine open set U , to a dg module $\mathcal{E}|_U$ over the Chevalley–Eilenberg algebra, and these modules are glued by filtered quasi-isomorphisms in a homotopy coherent way. For a general morphism of L_∞ algebroids $f : \mathcal{M} \rightarrow \mathcal{L}$ over bases Y and X , there is a pullback functor

$$f^* : \text{QCoh}([X/\mathcal{L}]) \rightarrow \text{QCoh}([Y/\mathcal{M}])$$

which is given on affine subsets $U \subset X$ and $V \subset Y$ by the usual formula

$$f^*\mathcal{E}|_V = \mathcal{E}|_U \otimes_{C^\bullet(\mathcal{L})} C^\bullet(\mathcal{M}).$$

3.2 The tangent and cotangent complexes

Suppose that \mathcal{L} is an L_∞ algebroid on a smooth affine manifold U and let $[U/\mathcal{L}]$ be the associated formal stack. Then $[U/\mathcal{L}]$ has tangent and cotangent complexes $\mathcal{T}_{[U/\mathcal{L}]}, \mathcal{T}_{[U/\mathcal{L}]}^\vee \in \mathbf{QCoh}([U/\mathcal{L}])$. In the algebraic setting, where U is a smooth scheme, these can be characterized by a universal property [63, Chapter I.4], as follows. (A similar statement is possible in the C^∞ and holomorphic contexts using C^∞ /analytic rings.)

For every $\mathcal{M} \in \mathbf{QCoh}([U/\mathcal{L}])$, the trivial square-zero extension $\mathcal{O}_{[U/\mathcal{L}]} \oplus \mathcal{M}$ of the structure sheaf gives an infinitesimal thickening

$$[U/\mathcal{L}] \rightarrow [U/\mathcal{L}][\mathcal{M}] := \mathrm{Spec}(\mathcal{O}_{[U/\mathcal{L}]} \oplus \mathcal{M})$$

The space of splittings of this map is isomorphic to $\mathrm{Hom}_{\mathbf{QCoh}([U/\mathcal{L}])}(\mathcal{T}_{[U/\mathcal{L}]}^\vee, \mathcal{M})$, which characterizes the cotangent complex $\mathcal{T}_{[U/\mathcal{L}]}^\vee$ up to equivalence. The tangent complex $\mathcal{T}_{[U/\mathcal{L}]}$ is then the dual of $\mathcal{T}_{[U/\mathcal{L}]}^\vee$.

Note that on the level of derived global functions, a splitting corresponds to a map of cdgas $\mathcal{O}([U/\mathcal{L}]) \rightarrow \mathcal{O}([U/\mathcal{L}]) \oplus \mathcal{M}$, which is equivalent to a derivation $\mathcal{O}([U/\mathcal{L}]) \rightarrow \mathcal{M}$ of $\mathcal{O}([U/\mathcal{L}])$ -modules. One can therefore establish the following:

Lemma 3.3. *Under the identification between objects of $\mathbf{QCoh}([U/\mathcal{L}])$ and quasi-coherent $\mathbf{C}^\bullet(\mathcal{L})$ -modules, $\mathcal{T}_{[U/\mathcal{L}]}^\vee$ and $\mathcal{T}_{[U/\mathcal{L}]}$ are identified with the filtered dg-modules of Kähler differentials and derivations of $\mathbf{C}^\bullet(\mathcal{L})$, respectively.*

We now explain how to work concretely with these dg modules, following the approach of [6, 25], which dealt with the case of Lie 1-algebroids.

By the universal property of Kähler differentials, there is a derivation given by the Rham derivative

$$d: \mathbf{C}^\bullet(\mathcal{L}) \rightarrow \mathcal{T}_{[U/\mathcal{L}]}^\vee,$$

and the differential on $\mathcal{T}_{[U/\mathcal{L}]}^\vee$ is determined by the fact that d is a morphism of complexes: $\delta d = d\delta$. Similarly, the differential on $\mathcal{T}_{[U/\mathcal{L}]}$ is defined by taking the commutator of derivations with the Chevalley–Eilenberg differential δ .

Viewing elements of \mathcal{L} as derivations on $\mathbf{C}^\bullet(\mathcal{L})$ by interior contraction gives an inclusion $\mathcal{L}[1] \rightarrow \mathcal{T}_{[U/\mathcal{L}]}$. Meanwhile, restricting derivations to the degree zero component of $\mathbf{C}^\bullet(\mathcal{L})$ gives a projection $\mathcal{T}_{[U/\mathcal{L}]} \rightarrow \mathcal{T}_U$. Extending these maps linearly, we obtain an exact sequence

$$0 \longrightarrow \mathbf{C}^\bullet(\mathcal{L}) \otimes_{\mathcal{O}(U)} \mathcal{L}[1] \longrightarrow \mathcal{T}_{[U/\mathcal{L}]} \longrightarrow \mathbf{C}^\bullet(\mathcal{L}) \otimes_{\mathcal{O}(U)} \mathcal{T}_U \longrightarrow 0 \quad (6)$$

of graded $\mathbf{C}^\bullet(\mathcal{L})$ -modules, which may be split by picking a connection on \mathcal{L} (i.e. an action $\mathcal{T}_U \times \mathcal{L} \rightarrow \mathcal{L}$ by degree zero derivations). We note that the submodule $\mathbf{C}^\bullet(\mathcal{L}) \otimes \mathcal{L}[1]$ is not a subcomplex; in general, it carries no natural differential.

Using (5) and (6) we may readily compute the pullback of the tangent complex to U :

Proposition 3.4. *The pullback of $\mathcal{T}_{[U/\mathcal{L}]}$ along $\pi: U \rightarrow [U/\mathcal{L}]$ is given by*

$$\pi^*\mathcal{T}_{[U/\mathcal{L}]} = \left(\cdots \xrightarrow{\delta} \mathcal{L}_1 \xrightarrow{\delta} \mathcal{L}_0 \xrightarrow{a} \mathcal{T}_U \right)$$

where \mathcal{T}_U sits in degree zero and \mathcal{L}_i sits in degree $-(i+1)$.

Remark 3.5. This result has the usual geometric interpretation: at a point $p \in U$, the zeroth cohomology of the fibre

$$H^0(\pi^*\mathcal{T}_U|_p) = T_p U / a(\mathcal{L}_0|_p)$$

is the normal space to the \mathcal{L} -orbit of p , i.e. the Zariski tangent space to the projection $\pi(p) \in [U/\mathcal{L}]$. Meanwhile, the negative cohomologies form a graded Lie algebra that represents the stabilizer of p under the \mathcal{L} -action. \square

Note that while the $C^\bullet(\mathcal{L})$ -modules $\mathcal{T}_{[U/\mathcal{L}]}^\vee$ and $\mathcal{T}_{[U/\mathcal{L}]}$ have a bigrading by weight and degree, the weight grading is not preserved by the differential (only the filtration). Nevertheless, it is convenient when one wants explicit formulae. Dualizing the sequence (6) of $C^\bullet(\mathcal{L})$ -modules and taking the weight- n piece, we obtain the sequence

$$0 \longrightarrow \wedge^n \mathcal{L}^\vee \otimes \mathcal{T}_U^\vee \longrightarrow (\mathcal{T}_{[U/\mathcal{L}]}^\vee)^{\bullet, n} \longrightarrow \wedge^{n-1} \mathcal{L}^\vee \otimes \mathcal{L}^\vee[-1] \longrightarrow 0 \quad (7)$$

which has the following differential-geometric interpretation [6, 46]:

Proposition 3.6. *The weight- n subspace of $\mathcal{T}_{[U/\mathcal{L}]}^\vee$ is canonically isomorphic to the $\mathcal{O}(U)$ -module of pairs $(\omega_n, \bar{\omega}_n)$, consisting of a first-order totally skew-symmetric polydifferential operator*

$$\omega_n: \underbrace{\mathcal{L} \times \cdots \times \mathcal{L}}_{n \text{ times}} \rightarrow \Omega^1(U)$$

and a tensor

$$\bar{\omega}_n \in \text{Sym}^{n-1}(\mathcal{L}^\vee[-1]) \otimes \mathcal{L}[-1]$$

related by the symbol equation

$$\omega_n(x_1, \dots, x_{n-1}, f x_n) = f \omega_n(x_1, \dots, x_n) + \bar{\omega}_n(x_1, \dots, x_{n-1}|x_n) \cdot df \quad (8)$$

Here $\bar{\omega}_n(x_1, \dots, x_{n-1}|x_n)$ denotes the canonical $\mathcal{O}(U)$ -linear pairing of $\bar{\omega}_n$ with $x_1 x_2 \cdots x_{n-1} \otimes x_n$.

Proof. The weight- n subspace is evidently spanned by monomials of the form $u_1 \cdots u_n df$ and $u_1 \cdots u_{n-1} du_n$ with $f \in \mathcal{O}(U)$ and $u_i \in \mathcal{L}^\vee$. We simply explain how to assign a pair $(\omega_n, \bar{\omega}_n)$ to such monomials, and leave to the reader the straightforward check that the map gives a well-defined isomorphism, e.g. that is compatible with the fundamental relation $d(fu) = (df)u + f(du)$ for Kähler differentials.

Firstly, for the monomial $u_1 \cdots u_n df$, we set $\bar{\omega}_n = 0$, so that the operator $\omega_n: \mathcal{L}^{\times n} \rightarrow \Omega^1(U)$ must be $\mathcal{O}(U)$ -multilinear. We then define ω_n by interpreting ω as a monomial in $\text{Sym}^n(\mathcal{L}^\vee[-1]) \otimes \Omega^1(U)$.

Secondly, for a monomial $\omega = u_1 \cdots u_{n-1} du_n$ we set

$$\bar{\omega}_n = u_1 \cdots u_{n-1} \otimes u_n$$

and define the operator ω_n by the formula

$$\omega_n(x_1, \dots, x_n) = \sum_{i=1}^n (-1)^\epsilon \langle u_1 \cdots u_{n-1}, x_1 \cdots \widehat{x}_i \cdots x_n \rangle d \langle u_n, x_i \rangle$$

where the sign ϵ is determined by the Koszul sign rule. \square

Given an element $\omega \in \mathcal{T}_{[U/\mathcal{L}]}^\vee$, we write $\omega = (\omega_n, \bar{\omega}_n)_{n \geq 0}$ to indicate the sequence of pairs obtained by applying [Proposition 3.6](#) to all of the weight components of ω . The differential and de Rham derivative can now be described explicitly in terms of brackets, extending the formulae for classical Lie algebroids [1]. We describe the idea and leave the verification of the formulae as an exercise to the reader.

Firstly, if $u \in \mathcal{O}([U/\mathcal{L}])$, we use the construction in [Proposition 3.6](#) to deduce that the weight- n part of its de Rham differential $du \in \mathcal{T}_{[U/\mathcal{L}]}^\vee$ is

$$\begin{aligned} (du)_n(x_1, \dots, x_n) &= d(u_n(x_1, \dots, x_n)) \\ \overline{(du)}_n(x_1, \dots, x_{n-1} | x_n) &= u_n(x_1, \dots, x_n). \end{aligned}$$

Secondly, the fact that d is a morphism of complexes results in the following formula for the differential of $\omega = (\omega_n, \bar{\omega}_n)$:

$$\begin{aligned} (\delta\omega)_n(x_1, \dots, x_n) &= (\delta_{\text{CE}}\omega)(x_1, \dots, x_n) \\ &\quad + \sum_{i=1}^n (-1)^\epsilon \mathcal{L}_{a_{x_i}} \omega_{n-1}(x_1, \dots, \widehat{x}_i, \dots, x_n) \\ \overline{(\delta\omega)}_n(x_1, \dots, x_{n-1} | x_n) &= (\delta_{\text{CE}}\bar{\omega})(x_1, \dots, x_{n-1} | x_n) \\ &\quad + \sum_{i=1}^{n-1} (-1)^\epsilon \mathcal{L}_{a_{x_i}} \bar{\omega}_{n-1}(x_1, \dots, \widehat{x}_i, \dots, x_{n-1} | x_n) \\ &\quad + (-1)^\epsilon \iota_{a_{x_n}} \omega_{n-1}(x_1, \dots, x_{n-1}). \end{aligned}$$

where $\delta_{\text{CE}}\omega$ is defined by the same formula as in (1), and $\delta_{\text{CE}}\bar{\omega}$ is defined similarly, but with the additional constraint that x_n always appears at the end, i.e. we sum over terms of the form $\bar{\omega}_j([x_{\sigma_1}, \dots, x_{\sigma_i}], x_{\sigma_{i+1}}, \dots, x_{\sigma_{n-1}} | x_n)$ and terms of the form $\bar{\omega}_j(x_{\sigma_1}, \dots, x_{\sigma_{j-1}} | [x_{\sigma_j}, \dots, x_{\sigma_{n-1}}, x_n])$.

3.3 Differential forms

Let \mathcal{L} be an L_∞ algebroid on an affine manifold U . Then the complex of p -forms on $[U/\mathcal{L}]$ is given by p th exterior power

$$\Omega_{[U/\mathcal{L}]}^p := \wedge^p \mathcal{T}_{[U/\mathcal{L}]}^\vee := \left(\text{Sym}^p \mathcal{T}_{[U/\mathcal{L}]}^\vee[-1] \right) [p] \in \text{QCoh}([U/\mathcal{L}])$$

which we can compute simply as the exterior power of dg $\mathbf{C}^\bullet(\mathcal{L})$ -modules. We write $\Omega_{[U/\mathcal{L}]}^{p,q}$ for the degree q part of this dg module (the p -forms of degree q).

In particular, the degree zero component is $\Omega_{[U/\mathcal{L}]}^{p,0} = \Omega_U^p$.

The de Rham differential on functions extends to forms of higher degree in the usual way, giving a differential

$$d: \Omega_{[U/\mathcal{L}]}^{p,\bullet} \rightarrow \Omega_{[U/\mathcal{L}]}^{p+1,\bullet},$$

so that $\Omega_{[U/\mathcal{L}]}^{\bullet,\bullet}$ is a bigraded bidifferential algebra, often called the Weil algebra of \mathcal{L} .

Definition 3.7. Let X be an arbitrary manifold, and let \mathcal{L} be an L_∞ algebroid on X . The **algebra of differential forms on $[X/\mathcal{L}]$** is the bigraded bidifferential algebra

$$\Omega^{\bullet,\bullet}([X/\mathcal{L}]) = \lim_{U \subset X} \Omega_{[U/\mathcal{L}]}^{\bullet,\bullet}$$

obtained by taking the limit over all affine open subsets of X .

If $X = U$ is affine, this simply recovers the description of $\Omega_{[U/\mathcal{L}]}^{\bullet,\bullet}$ as a dg module over $\mathcal{O}([U/\mathcal{L}]) \cong \mathbf{C}^\bullet(\mathcal{L})$, with the weight filtration forgotten. In general, one can think of $\Omega^{\bullet,\bullet}([X/\mathcal{L}])$ as a model for the Hodge diamond of $[X/\mathcal{L}]$; in particular, we have

$$\mathbb{R}^\bullet \Gamma([X/\mathcal{L}], \Omega_{[X/\mathcal{L}]}^p) \cong \Omega^{p,\bullet}([X/\mathcal{L}]). \quad (9)$$

Remark 3.8. Our definition of differential forms on $[X/\mathcal{L}]$ is different from the standard one, in which one takes the limit of de Rham complexes of all affine manifolds mapping to $[X/\mathcal{L}]$. We expect that these two approaches give the same result (namely, one needs to prove that the de Rham complex of $[U/\mathcal{L}]$ is equivalent to the Weil algebra of \mathcal{L}); a precise comparison will appear in a future publication. \square

Remark 3.9. Suppose that \mathcal{L} is a Lie 1-algebroid on a C^∞ manifold U and G_\bullet is the nerve of a Lie groupoid integrating \mathcal{L} . Then the p -forms on $[U/G]$ are described by the complex $\Omega^p(G_\bullet)$. There is a natural “van Est” map [6, 51] $\Omega^p(G_\bullet)$ to $\Omega^{p,\bullet}([U/\mathcal{L}])$, which was shown in [6] to induce an isomorphism on cohomology, under certain connectivity assumptions on the source fibres. (We remark that $\Omega^{i,j}([U/\mathcal{L}])$ was denoted by $W^{j,i}$ in that paper.) The isomorphism (9) is the analogue of this statement in the case of formal Lie ∞ -groupoids and it holds without hypotheses on \mathcal{L} , essentially because the source fibres of the formal groupoid are always contractible. \square

3.4 Closed forms

Unlike forms on a manifold, for which being closed is a property, in the derived or stacky settings, closure is extra data: we ask for the p -form ω to satisfy the equation $d\omega = 0$ only up to higher homotopies. This condition is phrased most succinctly by analogy with the Poincaré lemma. On a C^∞ manifold X , the sheaf of closed differential forms of degree at least p has a natural resolution by acyclic sheaves, given by its inclusion in the complex $(\Omega_X^{\geq p}, d)$. In the algebraic setting, or on a quotient $[X/\mathcal{L}]$, the Poincaré lemma will typically fail; a key insight of Pantev–Toën–Vaquié–Vezzosi [54], is that one may instead *define* the closed p -forms using the total complex

$$\Omega^{\geq p}([X/\mathcal{L}]) = \text{Tot} \left(\Omega^{p,\bullet}([X/\mathcal{L}]) \xrightarrow{d} \Omega^{p+1,\bullet}([X/\mathcal{L}]) \xrightarrow{d} \dots \right)$$

Definition 3.10. Let X be a manifold and \mathcal{L} an L_∞ algebroid on X . A **closed (p, q) -form on $[X/\mathcal{L}]$** is a cocycle

$$\omega \in \mathbb{Z}^{p+q} \Omega^{\geq p}([X/\mathcal{L}])$$

Thus a closed (p, q) -form consists of whole sequence of forms

$$\omega_p, \omega_{p+1}, \omega_{p+2}, \dots$$

where $\omega_{p+j} \in \Omega^{p+j, q-j}([X/\mathcal{L}])$, and these data satisfy the equations

$$\begin{aligned} \delta\omega_p &= 0 \\ d\omega_p + \delta\omega_{p+1} &= 0 \\ d\omega_{p+1} + \delta\omega_{p+2} &= 0 \\ &\vdots \end{aligned}$$

Coboundaries give a natural notion of (higher) homotopies between closed forms, so that closed forms naturally form a higher groupoid:

Definition 3.11. The **space of closed (p, q) -forms on $[X/\mathcal{L}]$** is the simplicial set $|\Omega^{\geq p, q}([X/\mathcal{L}])|$ assigned to the truncated complex $\tau^{\leq p+q} \Omega^{\geq p}([X/\mathcal{L}])$ by the Dold–Kan correspondence.

Similar to the one-form case described in [Proposition 3.6](#), one can describe arbitrary forms by decomposing them into weight components that are tensors and differential operators. But describing the space of closed (p, q) -forms in this way is cumbersome: for example, a closed $(2, 2)$ -form has nine distinct components, satisfying 15 different equations, and the higher homotopies add further complications.

Fortunately, most of this information is actually redundant. For example, [46, Proposition 4.12] implies that the projection $\pi: X \rightarrow [X/\mathcal{L}]$ induces an isomorphism on de Rham cohomology (which is, by definition, the space of

closed $(0,0)$ -forms). The intuitive reason is that π expresses $[X/\mathcal{L}]$ as a formal neighbourhood of X , so that the topologies are the same.

We now extend this result to give an efficient model for the closed p -forms. It is enough to describe the construction when \mathcal{L} is an L_∞ algebroid on an affine manifold U . Consider the canonical \mathbb{K} -linear Euler derivation $\xi \in \mathcal{T}_{[U/\mathcal{L}]}^0$, given by the endomorphism of $C^\bullet(\mathcal{L})$ that multiplies a homogeneous element by its degree. By duality, it defines a $C^\bullet(\mathcal{L})$ -linear map $\mathcal{T}_{[U/\mathcal{L}]} \rightarrow C^\bullet(\mathcal{L})$, which in turn extends uniquely to a $C^\bullet(\mathcal{L})$ -linear derivation

$$\iota_\xi : \Omega^{\bullet,\bullet}([U/\mathcal{L}]) \rightarrow \Omega^{\bullet-1,\bullet}([U/\mathcal{L}])$$

of the symmetric algebra $\Omega^{\bullet,\bullet}([U/\mathcal{L}]) = \text{Sym}_{C^\bullet(\mathcal{L})}(\mathcal{T}_{[U/\mathcal{L}]}^\vee[-1])$. This is the usual contraction operator of Cartan calculus (with appropriate Koszul signs).

Similarly, we have the Lie derivative

$$\mathcal{L}_\xi := [d, \iota_\xi] = d\iota_\xi + \iota_\xi d,$$

which is a derivation of bidegree $(0,0)$. One easily checks that $\Omega^{\bullet,q}([U/\mathcal{L}])$ is an eigenspace for \mathcal{L}_ξ with eigenvalue q . It follows immediately that the operator

$$h : \Omega^{\bullet,\bullet}([U/\mathcal{L}]) \rightarrow \Omega^{\bullet-1,\bullet}([U/\mathcal{L}]),$$

defined on homogeneous elements by the formula

$$h(\omega) = \begin{cases} \frac{1}{q} \iota_\xi \omega & \omega \in \Omega^{\bullet,q}([U/\mathcal{L}]), q > 0 \\ 0 & \omega \in \Omega^{\bullet,0}([U/\mathcal{L}]) = \Omega^\bullet(U), \end{cases}$$

is a contracting homotopy for $(\Omega^{\bullet,\bullet}([U/\mathcal{L}]), d)$. In particular, every strictly d-closed form ω of positive degree is d-exact, and has a canonical primitive given by $h\omega$.

Let us define the **complex of potentials**

$$\text{Pot}^{p-1,\bullet}([U/\mathcal{L}]) := \text{img } h \subset \Omega^{p-1,\bullet}([U/\mathcal{L}])$$

with differential defined by

$$\delta_{\text{Pot}} = h\delta d = -hd\delta = (dh - 1)\delta. \quad (10)$$

Then by the discussion above, h and d give mutually inverse isomorphisms between the complex of potentials, and the complex of strictly closed p -forms of positive degree.

There is also a natural **twisting map**

$$\text{Tw} : \Omega^{\geq p}(U) \rightarrow \text{Pot}^{p-1}([U/\mathcal{L}])[1]$$

defined as follows. Given an arbitrary element $G \in \Omega^{p+q}(U)$, we may contract it with the exterior power $\wedge^{q+1}a$ of the anchor $a \in \mathcal{L}_0^\vee \otimes \mathcal{T}_U$ to obtain the element

$$\text{Tw } G = \iota_{\wedge^{q+1}a} G \in \wedge^{q+1} \mathcal{L}_0^\vee \otimes \Omega^{p-1}(U)$$

which we view as a $(p-1)$ -form whose coefficient lies in $\wedge^{q+1} \mathcal{L}_0^\vee \subset \mathcal{O}([U/\mathcal{L}])$.

Definition 3.12. The *normalized complex of closed p -forms* is the complex

$$\Omega_{\text{cl}}^p([U/\mathcal{L}]) = \text{Pot}^{p-1}([U/\mathcal{L}]) \oplus \Omega^{\geq p}(U)$$

with differential given by

$$\delta_{\text{Tw}} = \begin{pmatrix} \delta_{\text{Pot}} & \text{Tw} \\ 0 & d \end{pmatrix} : \begin{array}{c} \text{Pot}^{p-1}([U/\mathcal{L}]) \\ \oplus \\ \Omega^{\geq p}(U) \end{array} \longrightarrow \begin{array}{c} \text{Pot}^{p-1}([U/\mathcal{L}]) \\ \oplus \\ \Omega^{\geq p}(U) \end{array}.$$

Thus the normalized complex fits in an exact sequence

$$0 \longrightarrow \text{Pot}^{p-1}([U/\mathcal{L}]) \longrightarrow \Omega_{\text{cl}}^p([U/\mathcal{L}]) \longrightarrow \Omega^{\geq p}(U) \longrightarrow 0.$$

Although the normalized complex contains no elements in $\Omega^{p+j, q-j}([U/\mathcal{L}])$ for $0 < j < q$, it still captures the full complexity of homotopy closed forms:

Theorem 3.13. *The normalized complex so-defined is, indeed, a complex. Moreover, there is a canonical homotopy equivalence*

$$\Omega^{\geq p}([U/\mathcal{L}]) \cong \Omega_{\text{cl}}^p([U/\mathcal{L}])$$

compatible with the projections to $\Omega^{\geq p}(U)$.

Proof. Consider first the case in which the differential and the L_∞ algebroid structure on \mathcal{L} are trivial, so that δ , δ_{Pot} and Tw are all zero. Thus the complex of normalized closed forms is the direct sum $(\text{Pot}^{p-1}([U/\mathcal{L}]) \oplus \Omega^{\geq p}(U), \delta_0)$. We have maps of complexes

$$(\Omega^{\geq p}([U/\mathcal{L}]), d) \xrightleftharpoons[i]{\phi} (\text{Pot}^{p-1}([U/\mathcal{L}]) \oplus \Omega^{\geq p}(U), \delta_0).$$

where the projection ϕ is given by the homotopy $h: \Omega^p([U/\mathcal{L}]) \rightarrow \text{Pot}^{p-1}(\mathcal{L})$ and the projection $\Omega^{\geq p}([U/\mathcal{L}]) \rightarrow \Omega^{\geq p}(U)$, while the inclusion i is given by $d: \text{Pot}^{p-1}(\mathcal{L}) \rightarrow \Omega^p([U/\mathcal{L}])$ and the inclusion $\Omega^{\geq p}(U) \rightarrow \Omega^{\geq p}([U/\mathcal{L}])$. A straightforward calculation shows that if we view h as a homotopy on the truncated complex $(\Omega^{\geq p}([U/\mathcal{L}]), d)$, then $hd + dh = 1 - i\phi$, giving the desired homotopy equivalence in this case.

We now consider the general case as a perturbation of the trivial one. That is, given a nontrivial L_∞ algebroid structure, we view the total differential $\delta + d$ on $\Omega^{\geq p}([U/\mathcal{L}])$ as a perturbation of the de Rham differential d , and apply the Homological Perturbation Lemma [13, 35] (see also [23]). Let

$$A = (1 + \delta h)^{-1} \delta = \delta - (\delta h) \delta + (\delta h)^2 \delta + \cdots,$$

which is well-defined by bidegree considerations. Then the formula

$$\delta' = \delta_0 + \phi A i$$

defines a differential on $\Omega_{\text{cl}}^p([U/\mathcal{L}])$ and there is a homotopy equivalence

$$(\Omega^{\geq p}([U/\mathcal{L}]), d + \delta) \xrightleftharpoons[i'=(1-hA)i]{\phi'=\phi(1-Ah)} (\text{Pot}^{p-1}([U/\mathcal{L}]) \oplus \Omega^{\geq p}(U), \delta').$$

Since h acts by zero on $\Omega^\bullet(U)$, it is clear that ϕ' intertwines the projections to $\Omega^{\geq p}(U)$. Thus to prove the theorem, it suffices to verify that δ' is precisely the twisted differential δ_{Tw} described above.

To this end, suppose $\alpha \in \text{Pot}^{p-1}([U/\mathcal{L}])$. Considering the bidegrees, we find

$$\begin{aligned} \delta' \alpha &= \delta_0 \alpha + \phi(1 + \delta h)^{-1} \delta i \alpha \\ &= 0 + h(1 - \delta h + \dots) \delta d \alpha \\ &= h \delta d \alpha \\ &= \delta_{\text{Pot}} \alpha \end{aligned}$$

so that δ and δ_{Tw} agree on $\text{Pot}^{p,\bullet}(\mathcal{L})$. Similarly, if $G \in \Omega^{p+q}(U)$, we find

$$\begin{aligned} \delta G &= dG + \phi(1 - \delta h + (\delta h)^2 + \dots) \delta i G \\ &= dG + h(-\delta h)^q \delta G \\ &= dG - (-h\delta)^{q+1} G, \end{aligned}$$

We claim that the operator $(-h\delta)^{q+1}$ is precisely the twisting map Tw . Indeed, considering the definition, it is enough by induction to show the operator $-h\delta$ acts on the subspace

$$\wedge^n \mathcal{L}_0^\vee \otimes \Omega^\bullet(U) \subset \mathcal{O}([U/\mathcal{L}]) \cdot \Omega(U) \subset \Omega([U/\mathcal{L}])$$

by the contraction $\frac{1}{n+1} i_a$, where $a \in \mathcal{L}_0^\vee \otimes \mathcal{T}_U$ is the anchor. But this is straightforward: given an element $u \in \wedge^n \mathcal{L}_0^\vee$ and $f_1, \dots, f_k \in \mathcal{O}(U)$, we compute

$$\begin{aligned} h\delta(u \, df_1 \cdots df_k) &= \frac{1}{n+1} \iota_\xi \left(\delta u \, df_1 \cdots df_k + \sum_{j=1}^k (-1)^{k+j} u \, d(\delta f_j) \, df_1 \cdots \widehat{df_j} \cdots df_k \right) \\ &= \frac{-1}{n+1} \sum_{j=1}^k (-1)^{j-1} (u \wedge a^\vee(df_j)) \cdot df_1 \cdots \widehat{df_j} \cdots df_k \\ &= -\frac{1}{n+1} \iota_a(u \, df_1 \cdots df_k) \end{aligned}$$

where we have used the Koszul sign rule and the identities

$$\iota_\xi df_i = 0 \quad \iota_\xi d(\delta f_j) = \delta f_j = a^\vee(df_j)$$

relating the differential δ , the anchor a and the Euler vector field ξ . □

The operator $\delta_{\text{Pot}} = dh\delta - \delta$ can be written explicitly in terms of its weight components by combining the formula for δ with an explicit formula for dh . We shall only need the action of dh on one-forms in the paper:

Example 3.14. Consider a monomial $\alpha = u_1 \cdots u_j dv$ with $u_1, \dots, u_j, v \in \mathcal{L}^\vee[-1]$. Using the fact that the contraction operator ι_ξ on has total degree -1 on $\Omega^{\bullet, \bullet}([X/\mathcal{L}])$, we see that

$$h\alpha = \frac{(-1)^{|\alpha| - |v|}}{|\alpha|} u_1 \cdots u_j \cdot v.$$

Applying d to this expression, and converting it back into operators as in [Proposition 3.6](#), we get the following formula for the action of dh on an arbitrary element $\alpha \in \Omega^1([U/\mathcal{L}])$, not just monomials:

$$\overline{(dh\alpha)}_n(x_1, \dots, x_{n-1}|x_n) = \frac{1}{|\alpha|} \sum_{i=1}^n (-1)^{\epsilon_{\bar{\alpha}_n}}(x_1, \dots, x_{i-1}, x_n, x_{i+1}, \dots, x_{n-1}|x_i)$$

and

$$(dh\alpha)_n(x_1, \dots, x_n) = \frac{1}{|\alpha|} d \left(\sum_{i=1}^n (-1)^{\epsilon_{\bar{\alpha}_n}}(x_1, \dots, x_{i-1}, x_n, x_{i+1}, \dots, x_{n-1}|x_i) \right)$$

where the Koszul sign is determined by treating $|$ as a degree one symbol.

Thus the effect of the operator dh on $\Omega^1([U/\mathcal{L}])$ is to apply an appropriate symmetrization to the tensorial part, and then adjust the operator component by an exact term that has the correct symbol. \square

4 Shifted symplectic structures

4.1 Shifted symplectic forms

We now recall the notions of shifted symplectic and Lagrangian structures [\[54\]](#), specialized to the case at hand. Given a cocycle $\omega \in Z^q(\Omega^{2, \bullet}([X/\mathcal{L}]))$, i.e. a global two-form of degree q , we obtain a morphism

$$\omega: \mathcal{T}_{[X/\mathcal{L}]} \rightarrow \mathcal{T}_{[X/\mathcal{L}]}^\vee[q],$$

by interior contraction. We say that ω is **nondegenerate** if this map is a quasi-isomorphism, i.e. an isomorphism in $\mathbf{QCoh}([X/\mathcal{L}])$.

Definition 4.1. Let X be a manifold, and let \mathcal{L} be an L_∞ algebroid on X . A **q -shifted symplectic structure on $[X/\mathcal{L}]$** is a closed $(2, q)$ -form

$$\omega \in Z^{2+q}\Omega^{\geq 2}([X/\mathcal{L}])$$

whose underlying two-form is nondegenerate in the above sense.

More explicitly, if $X = U$ is an affine manifold, then by [Lemma 3.2](#) and [Proposition 3.4](#), a two-form ω is nondegenerate if and only if its pullback induces

a quasi-isomorphism of complexes of vector bundles on U :

$$\begin{array}{ccccccc}
\pi^* \mathcal{T}_{[U/\mathcal{L}]} & \cdots \longrightarrow & \mathcal{L}_{q-1} & \xrightarrow{\delta} & \mathcal{L}_{q-2} & \xrightarrow{\delta} & \cdots \xrightarrow{a} \mathcal{T}_U \longrightarrow 0 \\
\downarrow \pi^* \omega & & \downarrow \omega & & \downarrow \omega & & \downarrow \omega \\
\pi^* \mathcal{T}_{[U/\mathcal{L}]}^\vee[q] & 0 \longrightarrow & \mathcal{T}_U^\vee & \xrightarrow{a^\vee} & \mathcal{L}_0^\vee & \xrightarrow{\delta} & \cdots \xrightarrow{\delta} \mathcal{L}_{q-1}^\vee \longrightarrow \cdots
\end{array} \tag{11}$$

Here the vertical maps are obtained by picking out appropriate tensorial components from the weight decomposition of ω .

Since the top complex is bounded on the right, while the bottom is bounded on the left, the quasi-isomorphism puts an obvious bound on \mathcal{L} . In particular, the natural truncation $\mathcal{L} \rightarrow \tau^{>(-q)} \mathcal{L}$ is a quasi-isomorphism of complexes of vector bundles, so by homotopy transfer ([Theorem 2.8](#)) we have the following:

Lemma 4.2. *Suppose that $[U/\mathcal{L}]$ admits a q -shifted symplectic structure for $q > 0$. Then \mathcal{L} is equivalent to a Lie q -algebroid.*

For a fixed L_∞ algebroid \mathcal{L} on a manifold X , the **space of q -shifted symplectic forms** is the full simplicial subset

$$\mathrm{Symp}_q([X/\mathcal{L}]) \subset |\Omega^{\geq 2,q}([X/\mathcal{L}])|$$

whose zero-simplices are q -shifted symplectic forms, where $|\Omega^{\geq 2,q}([X/\mathcal{L}])|$ is the space of closed two-form from [Definition 3.11](#).

Let $\mathrm{LA}(X) \subset L_\infty \mathrm{Algd}(X)$ be the maximal ∞ -groupoid, i.e. the subcategory with the same objects but with the non-invertible morphisms discarded. Symplectic structures may be pulled back along L_∞ quasi-isomorphisms, so that the assignment $\mathcal{L} \mapsto \mathrm{Symp}_q([X/\mathcal{L}])$ is functorial. Hence the simplicial sets $\mathrm{Symp}_q([X/\mathcal{L}])$ for varying \mathcal{L} may be assembled to give a single ∞ -groupoid:

Definition 4.3. The **space of q -shifted symplectic algebroids on X** is the simplicial set $\mathrm{SA}_q(X) \rightarrow \mathrm{LA}(X)$ obtained as the homotopy colimit of the functor $\mathrm{Symp}_q: \mathrm{LA}(X)^{op} \rightarrow \mathrm{SSet}$.

Thus an object of the ∞ -groupoid $\mathrm{SA}_q(X)$ is an L_∞ algebroid \mathcal{L} equipped with a shifted symplectic form ω . Meanwhile, a morphism $(\mathcal{L}, \omega) \rightarrow (\mathcal{L}', \omega')$ is an L_∞ -quasi-isomorphism $f: \mathcal{L} \rightarrow \mathcal{L}'$ together with a coboundary that trivializes the cocycle $f^* \omega' - \omega \in \Omega^{\geq 2,q}([X/\mathcal{L}])$, and similarly for higher homotopies.

We will use explicit models of the space $\mathrm{SA}_q(X)$ of q -shifted symplectic algebroids in certain cases. Suppose $q = 2$. Then by [Lemma 4.2](#) we are interested in the simplicial subset $\mathrm{LA}_2(X) \subset \mathrm{LA}(X)$ of Lie 2-algebroids. By [Theorem 2.7](#) $\mathrm{LA}_2(X)$ is the nerve of a strict 2-groupoid. Moreover, the space $\mathrm{Symp}_2([X/\mathcal{L}])$ is also the nerve of a strict 2-groupoid. A model of the corresponding homotopy colimit is then again a strict 2-groupoid given by the bicategorical Grothendieck construction, see e.g. [\[14\]](#).

4.2 Examples

We now give some simple examples of Lie algebroids equipped with shifted symplectic structures.

4.2.1 Zero-shifted symplectic algebroids

Arguing as in Lemma 4.2, we easily see that a zero-shifted symplectic algebroid must be quasi-isomorphic to a single vector bundle concentrated in degree zero, hence a classical Lie algebroid \mathcal{L} .

The only piece of data underlying a zero-shifted symplectic form on $[X/\mathcal{L}]$ is a two-form $B \in \Omega^{2,0}([X/\mathcal{L}]) = \Omega^2(X)$. By Theorem 3.13, the closure conditions amount to the equations

$$dB = 0 \in \Omega_X^3 \quad \text{Tw } B = \iota_a B = 0 \in \mathcal{L}^\vee \otimes \Omega_X^1,$$

and the nondegeneracy condition is that we have a quasi-isomorphism

$$\begin{array}{ccccc} \pi^* \mathcal{T}_{[X/\mathcal{L}]} & & \mathcal{L} & \xrightarrow{a} & \mathcal{T}_X & \longrightarrow & 0 \\ & \searrow \pi^* \omega & \downarrow & & \downarrow B & & \downarrow \\ \pi^* \mathcal{T}_{[X/\mathcal{L}]}^\vee & & 0 & \longrightarrow & \mathcal{T}_X^\vee & \xrightarrow{a^\vee} & \mathcal{L}^\vee. \end{array} \quad (12)$$

Considering the cohomology in degree one, we see that a^\vee must be surjective; equivalently, the anchor map embeds \mathcal{L} in \mathcal{T}_X as the tangent bundle of a regular foliation. Then the condition $\iota_a B = 0$ means that B descends to a two-form on the normal bundle $\mathcal{T}_X/\mathcal{L}$, which is forced to be nondegenerate by fact that (12) is an isomorphism on zeroth cohomology.

Finally, the condition $dB = 0$ means that B is closed in the normal directions, and also that it is invariant along the foliation. Thus we conclude that a zero-shifted symplectic Lie algebroid on X is simply a regular foliation of X , equipped with an invariant transverse symplectic structure, i.e. a classical symplectic structure on the leaf space $[X/\mathcal{L}]$.

4.2.2 Pullbacks along submersions

We thank the referee for pointing out the following construction, which one might view as a sort of higher analogue of the previous example. Suppose that $f : X \rightarrow Y$ is a submersion, and that (\mathcal{L}, ω) is a q -shifted symplectic L_∞ -algebroid over Y . Then we can form the shriek pullback L_∞ -algebroid $f^! \mathcal{L} = f^* \mathcal{L} \times_{f^* \mathcal{T}_Y} \mathcal{T}_X$ which sits in an exact sequence

$$0 \longrightarrow \mathcal{T}_f \longrightarrow f^! \mathcal{L} \longrightarrow \mathcal{L} \longrightarrow 0,$$

where \mathcal{T}_f is the relative tangent bundle of f . Using this sequence it is straightforward to check that $\mathcal{T}_{[X/f^! \mathcal{L}]} \cong f^* \mathcal{T}_{[Y/\mathcal{L}]}$, i.e. the map $[X/f^! \mathcal{L}] \rightarrow [Y/\mathcal{L}]$ is étale, so that the pullback of ω defines a q -shifted symplectic structure on $[X/f^! \mathcal{L}]$.

4.2.3 Transitive shifted symplectic algebroids

Now consider a Lie algebroid \mathcal{L} on a manifold X , and assume that \mathcal{L} that is *transitive*, i.e. its anchor map is surjective. Thus \mathcal{L} fits into an exact sequence

$$0 \longrightarrow \mathfrak{g} \longrightarrow \mathcal{L} \longrightarrow \mathcal{T}_X \longrightarrow 0,$$

where \mathfrak{g} is a vector bundle equipped with an \mathcal{O}_X -linear Lie bracket.

There is a natural adjoint action of \mathcal{L} on \mathfrak{g} , and hence \mathfrak{g} descends to a complex on $[X/\mathcal{L}]$, which we denote by $\mathfrak{g}_{[X/\mathcal{L}]}$. On an affine open subset U , it is represented by the dg module $C^\bullet(\mathcal{L}|_U) \otimes_{\mathcal{O}(U)} \mathfrak{g}$.

From the quasi-isomorphism

$$\pi^* \mathcal{T}_{[X/\mathcal{L}]} \cong \left(\mathcal{L} \longrightarrow \mathcal{T}_X \right) \cong \left(\mathfrak{g} \longrightarrow 0 \right) = \mathfrak{g}[1]$$

and the exact sequence (6), we see that the natural inclusion

$$\mathfrak{g}_{[X/\mathcal{L}]}[1] \rightarrow \mathcal{T}_{[X/\mathcal{L}]}$$

is a quasi-isomorphism of complexes on $[X/\mathcal{L}]$. (See also [25, Corollary 4].) Hence we have an identification $\Omega_{[X/\mathcal{L}]}^{\bullet, \bullet} \cong \text{Sym}(\mathfrak{g}_{[X/\mathcal{L}]}^\vee[-2])$.

Note that the quasi-isomorphism $\pi^* \mathcal{T}_{[X/\mathcal{L}]} \rightarrow \pi^* \mathcal{T}_{[X/\mathcal{L}]}^\vee[q]$ induced by a q -shifted symplectic structure gives an equivalence $\mathfrak{g}[1] \cong \mathfrak{g}^\vee[q-1]$, which forces $q = 2$ if $\mathfrak{g} \neq 0$. But the space of closed $(2, 2)$ -forms is given by the discrete set

$$H^0([X/\mathcal{L}], \text{Sym}^{\geq 2}(\mathfrak{g}_{[X/\mathcal{L}]}^\vee)) \cong H^0(X, \text{Sym}^2(\mathfrak{g}^\vee))^\mathcal{L},$$

of \mathcal{L} -invariant symmetric bilinear forms on \mathfrak{g} , so we have the following:

Proposition 4.4. *Let \mathcal{L} be a transitive Lie 1-algebroid on X , and suppose that the kernel $\mathfrak{g} \subset \mathcal{L}$ of its anchor is nontrivial. Then the only symplectic structures on $[X/\mathcal{L}]$ have shift two, and they are in bijective correspondence with nondegenerate \mathcal{L} -invariant symmetric bilinear forms on \mathfrak{g} .*

We recall that a transitive Lie 1-algebroid equipped with an invariant nondegenerate pairing on the kernel of its anchor map is often called a **quadratic Lie algebroid**.

The pullback of forms along the projection $X \rightarrow [X/\mathcal{L}]$ gives a natural map

$$H^0(X, \text{Sym}^2(\mathfrak{g}^\vee))^\mathcal{L} \rightarrow H^2(X, \Omega_X^{\geq 2}).$$

This connecting homomorphism may be computed by covering X with affine open subsets U_i on which the inclusion $\mathfrak{g} \subset \mathcal{L}$ can be split, and using the resulting projections $\mathcal{L}|_{U_i} \rightarrow \mathfrak{g}|_{U_i}$ to give an explicit cocycle representative in $\Omega^{\geq 2}([U_i/\mathcal{L}|_{U_i}])$. Comparing splittings on double and triple overlaps allows one to extend this to a cocycle on all of X .

We shall not give the details of this process here; let us simply state the result in a special case. Suppose that G is a Lie group equipped with a nondegenerate pairing $\langle -, - \rangle$ on its Lie algebra, and P is a principal G -bundle on X . Its Atiyah algebroid \mathcal{L} is an extension

$$0 \longrightarrow \text{ad } P \longrightarrow \mathcal{L} \longrightarrow \mathcal{T}_X \longrightarrow 0$$

and the kernel $\mathfrak{g} = \text{ad } P$ inherits an invariant nondegenerate pairing from $\langle -, - \rangle$, producing a two-shifted symplectic structure on $[X/\mathcal{L}]$. In this case, the pullback of the symplectic form gives the class in $H^2(X, \Omega_X^{\geq 2})$ associated with P and $\langle -, - \rangle$ by Chern–Weil theory, namely the first Pontryagin class [11, 57]. If we think of P as a map $X \rightarrow BG$, then $[X/\mathcal{L}]$ is a model for the formal neighbourhood of X in BG with its two-shifted symplectic structure [54, p. 299–300].

4.3 Isotropic and Lagrangian structures

Suppose that $f: (Y, \mathcal{M}) \rightarrow (X, \mathcal{L})$ is a morphism of manifolds equipped with L_∞ algebroids. Suppose further that $[X/\mathcal{L}]$ carries a q -shifted symplectic structure $\omega \in \text{Symp}_q([X/\mathcal{L}])$. Then we may ask if the induced morphism

$$f: [Y/\mathcal{M}] \rightarrow [X/\mathcal{L}]$$

defines a Lagrangian in $[X/\mathcal{L}]$. As with the definition of closed forms, the notion of Lagrangian corresponds to extra data on the map, rather than a property.

Definition 4.5. An *isotropic structure* on the map $f: [Y/\mathcal{M}] \rightarrow [X/\mathcal{L}]$ is a choice of coboundary for the cocycle $f^*\omega \in \Omega^{\geq 2}([Y/\mathcal{M}])$.

Picking out appropriate weight components of an isotropic structure, we obtain a null homotopy of the composite sequence

$$\mathcal{T}_{[Y/\mathcal{M}]} \longrightarrow f^*\mathcal{T}_{[X/\mathcal{L}]} \xrightarrow{\omega} \mathcal{T}_{[Y/\mathcal{M}]}^\vee[q] \quad (13)$$

and hence a morphism

$$\mathcal{N}_f \rightarrow \mathcal{T}_{[Y/\mathcal{M}]}^\vee[q]$$

where $\mathcal{N}_f = \text{Cone}(\mathcal{T}_{[Y/\mathcal{M}]} \rightarrow f^*\mathcal{T}_{[X/\mathcal{L}]})$ is the normal complex.

Definition 4.6. An isotropic structure is *Lagrangian* if the induced morphism $\mathcal{N}_f \rightarrow \mathcal{T}_{[Y/\mathcal{M}]}^\vee[q]$ is a quasi-isomorphism, or equivalently, (13) is a fibre sequence of complexes.

Example 4.7. If $X = \{*\}$ is a point and $\mathcal{L} = 0$, then $\mathcal{T}_{[X/\mathcal{L}]} = 0$, so that every q -shifted symplectic structure on $[X/\mathcal{L}]$ is equivalent to 0, and $\mathcal{N}_f \cong \mathcal{T}_{[Y/\mathcal{M}]}[1]$. A Lagrangian structure on the projection $[Y/\mathcal{M}] \rightarrow [X/\mathcal{L}] = \{*\}$ is therefore equivalent to a closed $(2, q-1)$ -form on $[Y/\mathcal{M}]$ that induces an equivalence $\mathcal{N}_f \cong \mathcal{T}_{[Y/\mathcal{M}]} \cong \mathcal{T}_{[Y/\mathcal{M}]}[q-1]$, i.e. a $(q-1)$ -shifted symplectic structure on $[Y/\mathcal{M}]$. \square

Example 4.8. Consider the case $X = Y$ and $\mathcal{M} = 0$, so that f is simply the quotient map

$$f = \pi: X \rightarrow [X/\mathcal{L}]$$

From the isomorphism

$$\pi^*\mathcal{T}_{[X/\mathcal{L}]} \cong \left(\cdots \longrightarrow \mathcal{L}_1 \longrightarrow \mathcal{L}_0 \xrightarrow{a} \mathcal{T}_U \right)$$

we see that $\mathcal{N}_\pi \cong \mathcal{L}[1]$, so that the isotropic structure induces a morphism $\mathcal{L}[1] \rightarrow \mathcal{T}_X^\vee[q]$. If the quotient map is Lagrangian, then $\mathcal{L} \cong \mathcal{T}_X^\vee[q-1]$. For $q > 1$ the only possibility is that \mathcal{L} is abelian (i.e. the anchor and brackets vanish), so this condition is quite restrictive. But as we recall in [Section 7.2](#), the case $q = 1$ is nontrivial: it gives the Lie algebroid \mathcal{T}_X^\vee associated to a Poisson structure. \square

Example 4.9. Suppose that \mathcal{L} carries a zero-shifted symplectic structure, so that it is defined by a regular foliation equipped with a transverse symplectic form $B \in \Omega^2(X)$ as in [Section 4.2.1](#). We claim that Lagrangians in $[X/\mathcal{L}]$ correspond to immersed Lagrangians in the leaf space of the foliation, in the sense of classical symplectic geometry.

Indeed, given a map $f: Y \rightarrow X$, consider a Lagrangian $[Y/\mathcal{M}] \rightarrow [X/\mathcal{L}]$ obtained by lifting f to a Lie algebroid morphism $\mathcal{M} \rightarrow \mathcal{L}$. Since the symplectic structure has degree zero, there is no room for homotopies between forms. Thus being Lagrangian is a condition, rather than extra data.

Computing the normal complex of $[Y/\mathcal{M}] \rightarrow [X/\mathcal{L}]$, we see that the map is Lagrangian if and only if B induces a quasi-isomorphism

$$\begin{array}{ccccccc} \pi^* \mathcal{N} & & \mathcal{M} & \longrightarrow & \mathcal{T}_Y & \longrightarrow & f^*(\mathcal{T}_X/\mathcal{L}) \longrightarrow 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow f^*B \\ \pi^* \mathcal{T}_{[Y/\mathcal{M}]}^\vee & & 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{T}_Y^\vee \longrightarrow \mathcal{M}^\vee. \end{array}$$

Considering the cohomology in degree 1 and -2 , we see that \mathcal{M} must embed in \mathcal{T}_Y as an involutive subbundle, giving a regular foliation of Y . Then, from the cohomology in degree -1 , we see that $\mathcal{T}_Y/\mathcal{M}$ embeds in $f^*(\mathcal{T}_X/\mathcal{L})$ as a subsheaf. Finally, considering the cohomology in degree zero, we see that this subsheaf must be a subbundle $\mathcal{T}_Y/\mathcal{M} \subset f^*(\mathcal{T}_X/\mathcal{L})$ that is maximally isotropic with respect to B . Hence the map $[Y/\mathcal{M}] \rightarrow [X/\mathcal{L}]$ is a Lagrangian immersion of the leaf spaces, as claimed. \square

Example 4.10. Let G be a Lie group equipped with an nondegenerate invariant bilinear form on its Lie algebra, and let $H \subset G$ be a closed subgroup whose corresponding Lie subalgebra is Lagrangian. If P is a principal G -bundle on X with Atiyah algebroid $\mathcal{L}(P)$, then $[X/\mathcal{L}(P)]$ is 2-shifted symplectic as in [Section 4.2.3](#). Moreover, if P admits a reduction of structure to a principal H -bundle P' , then natural inclusion of Atiyah algebroids $\mathcal{L}(P') \subset \mathcal{L}(P)$ gives a Lagrangian map $[X/\mathcal{L}(P')] \rightarrow [X/\mathcal{L}(P)]$. \square

5 Two-shifted symplectic forms

5.1 Twisted Courant algebroids

We now turn to the classification of shifted symplectic structures of low degree. The strategy is to use homotopy transfer and the normalized complex of closed two-forms to reduce the complicated data of a shifted symplectic L_∞ -algebroid to a normal form in terms of the following objects:

Definition 5.1 ([36, 48]). Let U be an affine manifold. A *twisted Courant algebroid* on U is a tuple $(\mathcal{E}, K, \langle -, - \rangle, \circ, a)$, where

- $K \in \Omega_{\text{cl}}^4(U)$ is a global closed 4-form

- \mathcal{E} is a locally free sheaf, i.e. a vector bundle
- $\langle -, - \rangle \in \text{Sym}^2(\mathcal{E}^\vee)$ is a nondegenerate symmetric bilinear pairing
- $a: \mathcal{E} \rightarrow \mathcal{T}_U$ is an \mathcal{O}_U -linear map, called the anchor, and
- $\llbracket -, - \rrbracket: \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E}$ is a bilinear operator, called the Courant–Dorfman bracket.

These data are subject to the following equations concerning their action on sections $x, y, z \in \mathcal{E}$:

$$\llbracket x, fy \rrbracket = f\llbracket x, y \rrbracket + (\mathcal{L}_a f)y \quad (14)$$

$$\llbracket x, x \rrbracket = \frac{1}{2}a^*d\langle x, x \rangle \quad (15)$$

$$\mathcal{L}_{ax}\langle y, z \rangle = \langle \llbracket x, y \rrbracket, z \rangle + \langle y, \llbracket x, z \rrbracket \rangle \quad (16)$$

$$\llbracket x, \llbracket y, z \rrbracket \rrbracket = \llbracket \llbracket x, y \rrbracket, z \rrbracket + \llbracket y, \llbracket x, z \rrbracket \rrbracket - \frac{1}{2}a^*\iota_{ax}\iota_{ay}\iota_{az}K, \quad (17)$$

where $a^*: \Omega_U^1 \rightarrow \mathcal{E}$ is the transpose of the anchor with respect to $\langle -, - \rangle$.

Definition 5.2. A twisted Courant algebroid is a *Courant algebroid* if its four-form is trivial: $K = 0$.

We shall often suppress the anchor, bracket and pairing in the notation, and simply say that \mathcal{E} or (\mathcal{E}, K) is a twisted Courant algebroid. If we wish to emphasize that a twisted Courant algebroid is a Courant algebroid, we may refer to it as “untwisted”.

Example 5.3. The original example of a Courant algebroid [20, 21, 29] is the bundle $\mathcal{E} = \mathcal{T}_U \oplus \mathcal{T}_U^\vee$. Its anchor is the projection to \mathcal{T}_U , and its pairing is the canonical one induced by the duality of \mathcal{T}_U and \mathcal{T}_U^\vee . Finally, the bracket is defined by

$$\llbracket x + \alpha, y + \beta \rrbracket = [x, y] + \mathcal{L}_x\beta - \iota_y d\alpha$$

where $x, y \in \mathcal{T}_U$ and $\alpha, \beta \in \mathcal{T}_U^\vee$. This Courant algebroid is called the *standard Courant algebroid on U* . \square

Example 5.4. Given a three-form $H \in \Omega^3(U)$, we can modify the bracket on the standard Courant algebroid by setting

$$\llbracket x + \alpha, y + \beta \rrbracket_H = \llbracket x + \alpha, y + \beta \rrbracket + \iota_x \iota_y H.$$

We then obtain a twisted Courant algebroid, with four-form $K = dH$. \square

We will give further examples in [Section 5.4](#). Twisted Courant algebroids on an affine manifold U naturally form a strict 2-groupoid $\text{TCA}(U)$, defined as follows:

Objects of $\text{TCA}(U)$ are twisted Courant algebroids (\mathcal{E}, K) on U .

1-Morphisms $(\mathcal{E}, K) \rightarrow (\mathcal{E}', K')$ are pairs (g, H) , where $g: \mathcal{E} \rightarrow \mathcal{E}'$ is an orthogonal bundle isomorphism that is compatible with the anchors, and $H \in \Omega^3(U)$ is a three-form that relates the brackets and four-forms:

$$\begin{aligned} g\llbracket x, y \rrbracket - \llbracket gx, gy \rrbracket' &= \frac{1}{2}a'^*\iota_{ax}\iota_{ay}H \\ K' - K &= dH \end{aligned}$$

2-morphisms $(g, H) \Rightarrow (\tilde{g}, \tilde{H})$ are two-forms $B \in \Omega^2(U)$ such that

$$\begin{aligned}\tilde{g} - g &= \frac{1}{2}a'^*Ba \\ \tilde{H} - H &= dB.\end{aligned}$$

Courant algebroids on U , in contrast, are more rigid: they do not admit differential forms as higher symmetries, and therefore form a 1-groupoid $\mathbf{CA}(U)$:

Objects of $\mathbf{CA}(U)$ are Courant algebroids \mathcal{E} on U

1-Morphisms $\mathcal{E} \rightarrow \mathcal{E}'$ in $\mathbf{CA}(U)$ are given by bundle isomorphisms that preserve the pairing, anchor and bracket.

For an inclusion $U' \subset U$ of affine manifolds, there are obvious restriction functors $\mathbf{TCA}(U) \rightarrow \mathbf{TCA}(U')$ and $\mathbf{CA}(U) \rightarrow \mathbf{CA}(U')$, obtained by pulling back bundles and forms. This allows us to define the space of (twisted) Courant algebroids on an arbitrary manifold X by gluing along open covers:

$$\mathbf{TCA}(X) = \lim_{U \subset X} \mathbf{TCA}(U) \quad \mathbf{CA}(X) = \lim_{U \subset X} \mathbf{CA}(U)$$

where the limit is taken over the category of all affine subsets of X .

For (untwisted) Courant algebroids, the result of such a gluing is evident: since the isomorphisms between Courant algebroids are strict isomorphisms of vector bundles that preserve all of the structure, an object of the 1-groupoid $\mathbf{CA}(X)$ is just a global vector bundle \mathcal{E} on X , equipped with an anchor, a pairing, and a bracket on its sheaf of sections, satisfying the axioms of [Definition 5.1](#) with $K = 0$. The notion of morphisms is the same as in the affine case.

But for twisted Courant algebroids, the situation is more complicated, due to the presence of 2-morphisms. With respect to an affine open covering $\{U_i\}$ of X , a twisted Courant algebroid \mathcal{E} on X is described by the following data:

- A twisted Courant algebroid (\mathcal{E}_i, K_i) on each affine open set U_i , as above
- A 1-morphism $(g_{ij}, H_{ij}): \mathcal{E}_i \rightarrow \mathcal{E}_j$ on every double overlap $U_i \cap U_j$
- A 2-morphism $B_{ijk}: (g_{ij}g_{jk}g_{ki}, H_{ij} + H_{jk} + H_{ki}) \Rightarrow (\text{id}_{\mathcal{E}_i}, 0)$ on every triple overlap $U_i \cap U_j \cap U_k$,

subject to an appropriate cocycle condition.

A key feature of twisted Courant algebroids is that the vector bundle gluing maps g_{ij} satisfy the twisted cocycle condition

$$g_{ij}g_{jk}g_{ki} = 1 + \frac{1}{2}a_i^*B_{ijk}a_i \in \text{Hom}_{U_{ijk}}(\mathcal{E}_i, \mathcal{E}_i). \quad (18)$$

The cocycle $B_{ijk} \in Z^2(X, \Omega_X^2)$ defines an Ω^2 -gerbe on X , and when this gerbe is nontrivial, \mathcal{E} will typically be a twisted bundle, rather than a global vector bundle in the classical sense.

More generally, the differential form data associated to a twisted Courant algebroid give a cocycle

$$(B_{ijk}, H_{ij}, K_i) \in Z^2(X, \Omega_X^{\geq 2})$$

in the hypercohomology of the truncated de Rham complex. We call its cohomology class the *twisting class of \mathcal{E}* :

$$[\mathcal{E}] = [(B_{ijk}, H_{ij}, K_i)] \in H^2(X, \Omega_X^{\geq 2}).$$

since it is the obstruction to finding an untwisted Courant algebroid that is equivalent to \mathcal{E} .

5.2 Classification of two-shifted symplectic structures

In this section we explain how to reduce an arbitrary two-shifted symplectic algebroid to a normal form, given in terms of twisted Courant algebroids. More precisely, we will prove the following

Theorem 5.5. *For any manifold X , there is a canonical equivalence*

$$\mathrm{SA}_2(X) \cong \mathrm{TCA}(X)$$

between the ∞ -groupoid of two-shifted symplectic L_∞ algebroids on X and the strict 2-groupoid of twisted Courant algebroids. Under this correspondence, the class in $H^2(X, \Omega_X^{\geq 2})$ determined by the pullback of a two-shifted symplectic structure agrees with the twisting class of the corresponding twisted Courant algebroid.

Considering the definitions, it is enough to prove the theorem for an arbitrary affine manifold U in a way that is compatible with restrictions to affine open subsets. More precisely, we will consider an auxiliary 2-groupoid $\mathrm{TCA}_{\mathrm{conn}}(U)$ and produce a canonical pair of equivalences

$$\begin{array}{ccc} & \mathrm{TCA}_{\mathrm{conn}}(U) & \\ \swarrow \sim & & \searrow \sim \\ \mathrm{SA}_2(U) & & \mathrm{TCA}(U), \end{array}$$

which will be functorial for restrictions to affine open subsets by construction. The strict 2-groupoid $\mathrm{TCA}_{\mathrm{conn}}(U)$ has a simple description: its objects are pairs (\mathcal{E}, ∇) , where \mathcal{E} is a twisted Courant algebroid and ∇ is a metric connection, i.e. a connection on the vector bundle \mathcal{E} that preserves the nondegenerate pairing. Morphisms in $\mathrm{TCA}_{\mathrm{conn}}(U)$ are morphisms of the underlying twisted Courant algebroids; the connections do not play a role.

The equivalence $\mathrm{TCA}_{\mathrm{conn}}(U) \rightarrow \mathrm{TCA}(U)$ is provided by the forgetful functor. Indeed, this functor is fully faithful by definition, and it is essentially surjective because every principal bundle on an affine manifold admits a connection. Thus the rest of the section is concerned with the construction of the equivalence $\mathrm{TCA}_{\mathrm{conn}}(U) \cong \mathrm{SA}_2(U)$.

5.2.1 Objects

We begin by describing the equivalence on the level of objects. Note that by [Lemma 4.2](#), an arbitrary two-shifted symplectic L_∞ algebroid is equivalent to a Lie 2-algebroid

$$\mathcal{L} = \left(\mathcal{L}_1 \longrightarrow \mathcal{L}_0 \right)$$

The symplectic structure on \mathcal{L} is determined by a potential $\beta \in \text{Pot}^{1,2}([U/\mathcal{L}])$ and a closed four-form $K \in \Omega_{\text{cl}}^4(U)$ such that $\delta_{\text{Pot}}\beta = \text{Tw} K$. Now β is a $(1,2)$ -form, and by degree considerations, its weight decomposition consists of operators

$$\phi = \beta_1: \mathcal{L}_1 \rightarrow \Omega^1(U) \quad \psi = \beta_2: \mathcal{L}_0 \times \mathcal{L}_0 \rightarrow \Omega^1(U)$$

and their symbols

$$\bar{\beta}_1 \in \mathcal{L}_1^\vee \quad Q = \bar{\beta}_2 \in \mathcal{L}_0^\vee \otimes \mathcal{L}_0^\vee$$

The condition $h\beta = 0$ for β to define an element of $\text{Pot}^{1,2}([U/\mathcal{L}])$ is equivalent to setting $\bar{\beta}_1 = 0$ identically, and requiring Q to be symmetric. Thus ϕ is $\mathcal{O}(U)$ -linear, while

$$\psi(x, fy) = f\psi(x, y) + Q(x, y)df$$

for $x, y \in \mathcal{L}_0$ and $f \in \mathcal{O}(U)$.

Applying the relation between the differential δ and the L_∞ algebroid structure, we see that closure equation $\delta_{\text{Pot}}\beta = \text{Tw} K$ is equivalent to the following four equations obtained from the weight and degree decomposition of $\delta_{\text{Pot}}\beta$:

$$\begin{aligned} & \mathcal{L}_{ax}\psi(y, z) + \psi(x, [y, z]) - \frac{1}{3}d(Q(x, [y, z]) + \iota_{ax}\psi(y, z)) + \circlearrowleft + \phi([x, y, z]) \\ &= \iota_{ax}\iota_{ay}\iota_{az}K \end{aligned} \quad (19)$$

$$\mathcal{L}_{ax}\phi(u) - \phi([x, u]) - \psi(x, \delta u) + dQ(x, \delta u) = 0 \quad (20)$$

$$\begin{aligned} & 3\mathcal{L}_{ay}Q(x, z) - 2Q(x, [y, z]) + \iota_{ax}\psi(y, z) - (x \leftrightarrow y) \\ & + 4Q([x, y], z) - 2\iota_{az}\psi(x, y) = 0 \end{aligned} \quad (21)$$

$$2Q(\delta u, x) + \iota_{ax}\phi(u) = 0 \quad (22)$$

where $x, y, z \in \mathcal{L}_0$ and $u \in \mathcal{L}_1$.

The last equation simply says that the pullback is a morphism of complexes

$$\begin{array}{ccccc} \pi^*\mathcal{T}_{[U/\mathcal{L}]} & & \mathcal{L}_1 & \xrightarrow{\delta} & \mathcal{L}_0 & \xrightarrow{a} & \mathcal{T}_U \\ \downarrow & & \downarrow \phi & & \downarrow \frac{1}{2}Q & & \downarrow \phi^\vee \\ \pi^*\mathcal{T}_{[U/\mathcal{L}]}^\vee[2] & & \mathcal{T}_U^\vee & \xrightarrow{a^\vee} & \mathcal{L}_0^\vee & \xrightarrow{\delta^\vee} & \mathcal{L}_1^\vee \end{array} \quad (23)$$

This diagram may be simplified in the following way:

Proposition 5.6. *Any two-shifted symplectic L_∞ algebroid is symplectically quasi-isomorphic to one for which the diagram (23) has the form*

$$\begin{array}{ccccc}
\pi^* \mathcal{T}_{[U/\mathcal{L}]} & & \mathcal{T}_U^\vee & \xrightarrow{\delta} & \mathcal{E} & \xrightarrow{a} & \mathcal{T}_U \\
\downarrow & & \parallel & & \downarrow \frac{1}{2} \langle -, - \rangle & & \parallel \\
\pi^* \mathcal{T}_{[U/\mathcal{L}]}^\vee[2] & & \mathcal{T}_U^\vee & \xrightarrow{a^\vee} & \mathcal{E}^\vee & \xrightarrow{\delta^\vee} & \mathcal{T}_U,
\end{array} \tag{24}$$

where $\langle -, - \rangle \in \text{Sym}^2(\mathcal{E}^\vee)$ is a nondegenerate symmetric bilinear form.

Proof. Consider the complex $\tilde{\mathcal{L}} = \mathcal{L} \oplus \mathcal{T}_U^\vee \oplus \mathcal{T}_U^\vee[1]$ with the differential twisted by ϕ and the identity $\mathcal{T}_U^\vee \rightarrow \mathcal{T}_U^\vee$ as follows:

$$\mathcal{T}_U^\vee \oplus \mathcal{L}_1 \xrightarrow{\begin{pmatrix} \text{id} & \phi \\ 0 & \delta \end{pmatrix}} \mathcal{T}_U^\vee \oplus \mathcal{L}_0$$

Let us also define $\tilde{\mathcal{E}} = \mathcal{L} \oplus \mathcal{T}_U^\vee$ with the differential twisted by ϕ . The natural projection $p: \tilde{\mathcal{L}} \rightarrow \tilde{\mathcal{E}}$ has a splitting $i: \tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{L}}$ given by

$$\begin{array}{ccc}
\mathcal{L}_1 & \longrightarrow & \mathcal{L}_0 \\
\downarrow (-\phi, \text{id}) & & \downarrow \text{id} \\
\mathcal{T}^\vee \oplus \mathcal{L}_1 & \longrightarrow & \mathcal{T}^\vee \oplus \mathcal{L}_0
\end{array}$$

making p into a deformation retract. Therefore, by Lemma 2.9 we obtain an L_∞ algebroid structure on $\tilde{\mathcal{L}}$; moreover, pulling back the two-shifted symplectic structure on \mathcal{L} along p we obtain a two-shifted symplectic structure $\tilde{\omega}$ on $\tilde{\mathcal{L}}$ of the following shape:

$$\begin{array}{ccccc}
\mathcal{T}_U^\vee \oplus \mathcal{L}_1 & \longrightarrow & \mathcal{T}_U^\vee \oplus \mathcal{L}_0 & \longrightarrow & \mathcal{T}_U \\
\downarrow (0, \phi) & & \downarrow \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix} & & \downarrow (0, \phi) \\
\mathcal{T}_U^\vee & \longrightarrow & \mathcal{T}_U \oplus \mathcal{L}_0^\vee & \longrightarrow & \mathcal{T}_U \oplus \mathcal{L}_1^\vee
\end{array}$$

We have a subspace $\mathcal{T}_U \otimes \mathcal{T}_U^\vee \subset \tilde{\mathcal{L}}_0^\vee \otimes \mathcal{T}_U^\vee \subset \text{Pot}^{1,1}(\tilde{\mathcal{L}})$ and it contains a canonical element τ corresponding to the identity. The form $\tilde{\omega} + \delta_{\text{Tw}} \tau$ is still nondegenerate and it has the following shape:

$$\begin{array}{ccccc}
\mathcal{T}_U^\vee \oplus \mathcal{L}_1 & \longrightarrow & \mathcal{T}_U^\vee \oplus \mathcal{L}_0 & \longrightarrow & \mathcal{T}_U \\
\downarrow (\text{id}, 0) & & \downarrow \frac{1}{2} \begin{pmatrix} 0 & a \\ a^\vee & Q \end{pmatrix} & & \downarrow (\text{id}, 0) \\
\mathcal{T}_U^\vee & \longrightarrow & \mathcal{T}_U \oplus \mathcal{L}_0^\vee & \longrightarrow & \mathcal{T}_U \oplus \mathcal{L}_1^\vee
\end{array}$$

The nondegeneracy of the two-shifted symplectic structure on \mathcal{L} is now equivalent to the morphism

$$\begin{pmatrix} 0 & a \\ a^\vee & Q \end{pmatrix}: \tilde{\mathcal{E}} \longrightarrow \tilde{\mathcal{E}}^\vee$$

being a quasi-isomorphism. But $\tilde{\mathcal{E}}$ is a complex of vector bundles concentrated in non-positive degrees, so the projection $\tilde{\mathcal{E}} \rightarrow \mathcal{H}^0(\tilde{\mathcal{E}})$ is also a quasi-isomorphism. Therefore, we can replace $\tilde{\mathcal{E}}$ by its cohomology $\mathcal{E} = \mathcal{H}^0(\tilde{\mathcal{E}})$ on which the pairing is strictly nondegenerate and the claim follows. \square

We say that a two-shifted symplectic algebroid is in **Courant form** if it has the form described in Proposition 5.6. Given an algebroid in Courant form, we may use the nondegeneracy of the pairing to define a connection $\nabla: \mathcal{E} \rightarrow \Omega^1(U) \otimes \mathcal{E}$ and a bracket $\llbracket -, - \rrbracket: \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E}$ by the formulae

$$\begin{aligned} \langle \nabla x, y \rangle &= \frac{1}{2}(\mathrm{d}\langle x, y \rangle - \psi(x, y)) \\ \langle \llbracket x, y \rrbracket, z \rangle &= \langle [x, y], z \rangle + \langle \nabla_{az} x, y \rangle \\ &= \langle [x, y], z \rangle + \frac{1}{2}\mathcal{L}_{az} \langle x, y \rangle - \frac{1}{2}\iota_{az} \psi(x, y) \end{aligned} \tag{25}$$

or equivalently

$$\llbracket x, y \rrbracket = [x, y] + \frac{1}{2}a_{\mathcal{E}}^* \mathrm{d} \langle x, y \rangle - \frac{1}{2}a_{\mathcal{E}}^* \psi(x, y). \tag{26}$$

Because of the skew-symmetry of ψ , the connection ∇ is automatically metric, i.e. it satisfies the equation

$$\mathrm{d}\langle x, y \rangle = \langle \nabla x, y \rangle + \langle x, \nabla y \rangle.$$

The following result then describes the equivalence $\mathrm{TCA}_{\mathrm{conn}}(U) \rightarrow \mathrm{SA}_2(U)$ on the level of objects:

Proposition 5.7. *The formulae (25) and (26) give a bijective correspondence between shifted symplectic L_∞ algebroids in Courant form and twisted Courant algebroids equipped with a metric connection.*

Proof. For an algebroid in Courant form, we have that $\phi = \mathrm{id}$. Thus the closure conditions (19), (20) and (22) uniquely determine the triple bracket $\mathcal{L}_0 \times \mathcal{L}_0 \times \mathcal{L}_0 \rightarrow \mathcal{L}_1$, the binary bracket $\mathcal{L}_0 \times \mathcal{L}_1 \rightarrow \mathcal{L}_1$ and the differential δ in terms of the remaining data. It is therefore sufficient to see that the axioms for a twisted Courant algebroid are the same as the remaining equations for the symplectic L_∞ algebroid structure.

Indeed, axiom (14) for a twisted Courant algebroid is equivalent to the Leibniz rule for the L_∞ bracket $[-, -]: \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E}$, axiom (15) is equivalent to the antisymmetry of the bracket $[-, -]$, and axiom (17) is equivalent to the Jacobi rule for the L_∞ brackets on \mathcal{L} . Finally, axiom (16) is equivalent to the equation

$$\mathcal{L}_{a(x)} Q(y, z) = Q([x, y], z) + \frac{1}{2}\mathcal{L}_{a(y)} Q(x, z) - \frac{1}{2}\iota_{a(y)} \psi(x, z) + (y \leftrightarrow z),$$

which is the symmetrization of the remaining closure equation (21). Conversely, if (16) is satisfied, the left-hand side of (21) is completely antisymmetric, but its antisymmetrization is obviously zero. Therefore, axiom (16) is equivalent to (21). \square

5.2.2 1-morphisms

Suppose we are given a pair (\mathcal{L}, ω) and (\mathcal{L}', ω') of symplectic algebroids in Courant form, corresponding to twisted Courant algebroids (\mathcal{E}, K) and (\mathcal{E}', K') . An L_∞ morphism $g: \mathcal{L} \rightarrow \mathcal{L}'$ consists of a quasi-isomorphism of complexes

$$g: \mathcal{L} \rightarrow \mathcal{L}'$$

that preserves the anchors, and a map

$$\tilde{g}: \wedge^2 \mathcal{E} \rightarrow \mathcal{T}_U^\vee = \mathcal{L}'_1,$$

satisfying the L_∞ morphism equations

$$g[x, y] - [gx, gy] = \delta \tilde{g}(x, y) \quad (27)$$

$$g[x, u] - [gx, gu] = \tilde{g}(\xi, \delta u) \quad (28)$$

$$\begin{aligned} g[x, y, z] - [gx, gy, gz] &= -\tilde{g}([x, y], z) + \tilde{g}([x, z], y) - \tilde{g}([y, z], x) \\ &\quad + [\tilde{g}(x, y), gz] - [\tilde{g}(x, z), gy] + [\tilde{g}(y, z), gx] \end{aligned} \quad (29)$$

for $x, y, z \in \mathcal{E}$ and $u \in \mathcal{T}_U^\vee$.

To extend such a quasi-isomorphism to a symplectic equivalence, we must include a homotopy of closed two-forms, given by an element of

$$\Omega_{\text{cl}}^{2,1}([U/\mathcal{L}]) = \text{Pot}^{1,1}([U/\mathcal{L}]) \oplus \Omega^3(U).$$

The elements of $\text{Pot}^{1,1}([U/\mathcal{L}])$ are $(1, 1)$ -forms in the image of h . Considering the weight decomposition, it is easy to see that in fact $\text{Pot}^{1,1}([U/\mathcal{L}]) \cong \mathcal{E}^\vee \otimes \Omega^1(U)$. The homotopy then consists of elements

$$\tau \in \mathcal{E}^\vee \otimes \Omega^1(U) \quad H \in \Omega^3(U)$$

satisfying the homotopy equation $g^* \omega' - \omega = \delta_{\text{Tw}}(\tau + H)$, which gives the system

$$K' - K = dH \quad (30)$$

$$gu - u = -\tau(\delta u) \quad (31)$$

$$\langle gx, gy \rangle - \langle x, y \rangle = \frac{1}{2}(\iota_{ax} \tau y + \iota_{ay} \tau x) \quad (32)$$

$$\begin{aligned} -\tilde{g}(x, y) + \psi'(gx, gy) - \psi(x, y) &= \tau[x, y] - \mathcal{L}_{ax} \tau y + \mathcal{L}_{ay} \tau x \\ &\quad + \frac{1}{2} d(\iota_{ax} \tau y - \iota_{ay} \tau x) + \iota_{ax} \iota_{ay} H \end{aligned} \quad (33)$$

Observe that the equation (28) follows from (33). Similarly, equation (29) follows from the definition of the triple bracket $[-, -, -]$ given by (19). Equation (31) determines the morphism $g: \mathcal{L} \rightarrow \mathcal{L}'$ in degree -1 and equation (33) determines $\tilde{g}(x, y)$.

We conclude that a 1-morphism in $\text{SA}_2(X)$ is uniquely determined by the triple (g, τ, H) , where $g: \mathcal{E} \rightarrow \mathcal{E}'$ is bundle map, $\tau \in \mathcal{E}^\vee \otimes \Omega^1(U)$ and $H \in \Omega^3(U)$ satisfy the equations (27), (30) and (32).

We say that a 1-morphism is in **Courant form** if $\tau = 0$. In this case, the equations reduce to the equations for (g, H) to give a 1-morphism of twisted Courant algebroids $(\mathcal{E}, K) \rightarrow (\mathcal{E}', K')$. In this way, we define the functor $\text{TCA}_{\text{conn}}(U) \rightarrow \text{SA}_2(U)$ on the level of 1-morphisms.

5.2.3 2-morphisms

Finally, suppose we are given a pair of 1-morphisms $f_i: \mathcal{L} \rightarrow \mathcal{L}'$ for $i = 1, 2$ determined by the data $g_i: \mathcal{E} \rightarrow \mathcal{E}'$, $\tau_i \in \mathcal{E}^\vee \otimes \Omega^1(U)$ and $H_i \in \Omega^3(U)$ as above.

A 2-morphism $f_1 \Rightarrow f_2$ in $\mathbf{SA}_2(U)$ consists of a homotopy operator on the complexes, i.e. an \mathcal{O}_U -linear map

$$h: \mathcal{E} \rightarrow \mathcal{T}_U^\vee = \mathcal{L}'_1$$

and a form

$$B \in \Omega_{\text{cl}}^{2,0}([U/\mathcal{L}]) = \Omega^2(U)$$

We require $(\delta + d)B$ to equal the difference of the 2-form data appearing in f_i , giving the equations

$$H_2 - H_1 = dB \quad \quad \quad \tau_2 x - \tau_1 x = -\iota_{ax} B - hx \quad (34)$$

Evidently, this equation uniquely determines h from the rest of the data.

The L_∞ homotopy equations read

$$g_2 x - g_1 x = -\frac{1}{2} a^* h x \quad (35)$$

$$g_2 u - g_1 u = -\frac{1}{2} h(a'^* u) \quad (36)$$

$$\tilde{g}_2(x, y) - \tilde{g}_1(x, y) = h[x, y] - [hx, g_1 y] - [g_1 x, hy]. \quad (37)$$

It is easy to see that equation (36) follows from equations (31) and (34). Similarly, equation (37) follows from equations (33) and (34). We may now complete the proof of the main result:

Proof of Theorem 5.5. Considering the computations at the level of objects, morphisms and two-morphisms, we have evidently produced a 2-functor

$$\mathbf{TCA}_{\text{conn}}(U) \rightarrow \mathbf{SA}_2(U).$$

By Proposition 5.6 and Proposition 5.7, this 2-functor is essentially surjective, so we just have to show that it is fully faithful, i.e. that for two twisted Courant algebroids $\mathcal{E}_1, \mathcal{E}_2$ and the corresponding two-shifted symplectic L_∞ algebroids $\mathcal{L}_1, \mathcal{L}_2$, the functor

$$\text{Hom}_{\mathbf{TCA}_{\text{conn}}(U)}(\mathcal{E}_1, \mathcal{E}_2) \rightarrow \text{Hom}_{\mathbf{SA}_2(U)}(\mathcal{L}_1, \mathcal{L}_2)$$

is an equivalence of 1-groupoids. It is clearly fully faithful since the 2-morphisms in both $\mathbf{TCA}_{\text{conn}}(U)$ and $\mathbf{SA}_2(U)$ are determined by a 2-form B satisfying the same set of equations. To see that it is essentially surjective, we must show that any one-morphism in $\text{Hom}_{\mathbf{SA}_2(U)}(\mathcal{L}_1, \mathcal{L}_2)$ is equivalent to one in Courant form (i.e. with $\tau = 0$). But this follows immediately from (34) and (35). \square

5.3 Classification of isotropic quotients

Let X be a manifold, and let $\mathbf{SA}_2^{\text{iso}}(X)$ be ∞ -groupoid of two-shifted symplectic algebroids (\mathcal{L}, ω) on X , equipped with an isotropic structure on the quotient map

$$X \rightarrow [X/\mathcal{L}]$$

Then we immediately have the following result.

Proposition 5.8. *For any manifold X , the ∞ -groupoid $\mathbf{SA}_2^{\text{iso}}(X)$ is equivalent to the 1-groupoid $\mathbf{CA}(X)$ of Courant algebroids. In particular, a twisted Courant algebroid is equivalent to an untwisted Courant algebroid if and only if its twisting class vanishes.*

Proof. It is enough to establish the claim for affine manifolds U . The data of a two-shifted symplectic structure on an L_∞ algebroid \mathcal{L} and an isotropic structure on $U \rightarrow [U/\mathcal{L}]$ is equivalent to the data of a non-degenerate closed $(2, 2)$ -form in the homotopy fibre of the projection

$$\Omega_{\text{cl}}^2([U/\mathcal{L}]) \rightarrow \Omega^{\geq 2}(U).$$

But by construction, this morphism is surjective; hence the homotopy fibre is equivalent to the strict fibre. It follows that the ∞ -groupoid $\mathbf{SA}_2^{\text{iso}}(U)$ is equivalent to the subgroupoid of $\mathbf{SA}_2(U)$ in which we set all differential forms that live purely on U to zero. This corresponds to setting $K = 0$ on the level of objects, $H = 0$ on the level of morphisms and $B = 0$ on the level of 2-morphisms. Via [Theorem 5.5](#), this subgroupoid is naturally identified with the subgroupoid $\mathbf{CA}(U) \subset \mathbf{TCA}(U)$ of untwisted Courant algebroids. \square

Remark 5.9. Note that by [Example 4.8](#) the projection $X \rightarrow [X/\mathcal{L}]$ is Lagrangian iff $\mathcal{E} = 0$. \square

As a special case, recall that a Courant algebroid is *exact* if the anchor and its dual give an exact sequence of vector bundles

$$0 \longrightarrow \mathcal{T}_X^\vee \xrightarrow{a^*} \mathcal{E} \xrightarrow{a} \mathcal{T}_X \longrightarrow 0.$$

Equivalently, the anchor $\mathcal{L} \rightarrow \mathcal{T}_X$ of the corresponding two-shifted symplectic algebroid is a quasi-isomorphism.

Thus exact Courant algebroids on X are the same thing as two-shifted symplectic structures on $[X/\mathcal{T}_X]$ together with an isotropic structure on the quotient $X \rightarrow [X/\mathcal{T}_X]$. But the tangent complex of $[X/\mathcal{T}_X]$ is contractible, and hence the only symplectic structure is the trivial one. Nevertheless, isotropic structures can be nontrivial: they are primitives for the zero element in $Z^2(X, \Omega_X^{\geq 2})$, i.e. cocycles in $Z^1(X, \Omega_X^{\geq 2})$, and equivalences are provided by coboundaries. In this way, we obtain a symplectic interpretation of Ševera's cohomological classification of exact Courant algebroids:

Corollary 5.10 ([12, 57, 59]). *The stack of exact Courant algebroids is equivalent to the stack $\Omega^{\geq 2}[1]$ of 1-shifted closed two-forms. Thus an exact Courant algebroid \mathcal{E} on a manifold X is determined up to isomorphism by a class in $H^1(X, \Omega_X^{\geq 2})$, called its “Severa class”, and the group of base-fixing automorphisms of \mathcal{E} is the additive group of global closed two-forms on X .*

5.4 Examples

5.4.1 The transitive case

A twisted Courant algebroid is **transitive** if its anchor map is surjective. This is evidently equivalent to requiring that the L_∞ algebroid $\mathcal{T}_X \rightarrow \mathcal{E}$ is a Lie 1-algebroid $\mathcal{E}/\mathcal{T}_X$. Thus transitive twisted Courant algebroids are the same thing as classical Lie algebroids \mathcal{L} equipped with two-shifted symplectic structures as described in Section 4.2.3. From Proposition 5.8, we immediately obtain the following result.

Corollary 5.11 ([11, 57]). *A quadratic Lie algebroid can be extended to a transitive Courant algebroid if and only if its first Pontryagin class vanishes.*

If $\mathfrak{g} \subset \mathcal{L}$ is the kernel of the anchor map and $U \subset X$ is an affine open subset, we can split $\mathcal{L}|_U = \mathcal{T}_U \oplus \mathfrak{g}$. The twisted Courant algebroid is then given locally by $\mathcal{E}|_U = \mathcal{T}_U \oplus \mathfrak{g} \oplus \mathcal{T}_U^\vee$, equipped with the obvious pairing. The bracket involves the Courant bracket on $\mathcal{T}_U \oplus \mathcal{T}_U^\vee$, the Lie bracket on \mathfrak{g} and the curvature of the splitting $\mathcal{T}_U \rightarrow \mathcal{L}$. We refer the reader to [11, 57] for the explicit formulae. See [70] for an explicit example of such a twisted Courant algebroid. We also refer to [60] for a relationship between transitive Courant algebroids and string structures.

5.4.2 Atiyah algebroids of perfect complexes

Recall that a perfect complex on X is a complex of quasi-coherent sheaves that is locally equivalent to a finite complex of finite rank vector bundles. The classifying stack of perfect complexes carries a two-shifted symplectic structure [54]. So by analogy with the case G -bundles discussed in Section 5.4.1, we expect the Atiyah algebroid of a perfect complex $\mathcal{F} = \mathcal{F}^\bullet \in \text{Perf}(X)$ to carry a two-shifted symplectic structure modeling the formal completion of Perf along the map $\mathcal{F} : X \rightarrow \text{Perf}$. We refer the reader to [37, Section 10.1] for an introduction to Atiyah algebroids of perfect complexes.

The Atiyah algebroid $\mathcal{L} = \mathcal{L}(\mathcal{F})$ sits in an exact triangle

$$\mathbb{R}\mathcal{E}nd(\mathcal{F}) \longrightarrow \mathcal{L} \longrightarrow \mathcal{T}_X \longrightarrow \mathbb{R}\mathcal{E}nd(\mathcal{F})[1],$$

with the derived endomorphisms of \mathcal{F} . Thus $\mathcal{T}_{[X/\mathcal{L}]}$ is isomorphic to the complex on $[X/\mathcal{L}]$ determined by the natural action of \mathcal{L} on $\mathbb{R}\mathcal{E}nd(\mathcal{F})[1]$. This complex carries the nondegenerate trace pairing, giving a two-shifted symplectic structure ω on $[X/\mathcal{A}]$ with pullback

$$[\pi^*\omega] = \text{ch}_2(\mathcal{F}) \in H^2(X, \Omega_X^{\geq 2}),$$

the degree-two part of the Chern character.

But there is a subtlety: the Atiyah algebroid is not, in general, an L_∞ algebroid: it will typically have cohomology in positive degrees, and thus be a *derived* L_∞ algebroid. This corresponds geometrically to the fact that the stack \mathbf{Perf} is quite singular.

So in order to obtain (underived) twisted Courant algebroids, we must rule out cohomology in positive degrees. The long exact sequence in cohomology gives

$$0 \rightarrow \mathcal{E}xt^0(\mathcal{F}, \mathcal{F}) \rightarrow \mathcal{H}^0(\mathcal{L}) \rightarrow \mathcal{T}_X \rightarrow \mathcal{E}xt^1(\mathcal{F}, \mathcal{F}) \rightarrow \mathcal{H}^1(\mathcal{L}) \rightarrow 0 \quad (38)$$

and isomorphisms $\mathcal{H}^i(\mathcal{L}) \cong \mathcal{E}xt^i(\mathcal{F}, \mathcal{F})$ for $i \neq 0, 1$. We remark that $\mathcal{E}xt^1(\mathcal{F}, \mathcal{F})$ is the sheaf of infinitesimal deformations of \mathcal{F} and the map $\mathcal{T}_X \rightarrow \mathcal{E}xt^1(\mathcal{F}, \mathcal{F})$ gives the infinitesimal deformations that arise by pulling back \mathcal{F} along flows of vector fields. From the exact sequence, we obtain the following

Proposition 5.12. *Let \mathcal{F} be a perfect complex on X . Then the Atiyah algebroid of \mathcal{F} is an L_∞ algebroid if and only if*

$$\mathcal{E}xt^i(\mathcal{F}, \mathcal{F}) = 0 \text{ for } i > 1$$

and the natural map

$$\mathcal{T}_X \rightarrow \mathcal{E}xt^1(\mathcal{F}, \mathcal{F}),$$

is surjective. In this case, there is a canonical twisted Courant algebroid associated to \mathcal{F} , whose twisting class is $\text{ch}_2(\mathcal{F})$.

Note that, while the Atiyah algebroid of a principal bundle is transitive, this is not the case for general perfect complexes. Since the Atiyah algebroid represents the infinitesimal symmetries of the complex, its orbits are related to the stratification of X by the singularities of \mathcal{F} .

5.4.3 Codimension-two cycles

In the complex analytic or algebraic settings, the appearance of $H^2(X, \Omega_X^2)$ suggests a link between twisted Courant algebroids and codimension-two cycles. Indeed, this is a special case of the previous example, as we now explain.

Suppose that X is a complex manifold or smooth algebraic variety, and let $Y \subset X$ be a smooth subvariety of pure codimension two. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf of Y . Then \mathcal{I} is a coherent sheaf on X , and since X is smooth, \mathcal{I} is perfect. We recall the standard canonical isomorphisms

$$\mathcal{E}xt^i(\mathcal{I}, \mathcal{I}) \cong \begin{cases} \mathcal{O}_X & i = 0 \\ \mathcal{N}_Y & i = 1 \\ 0 & \text{otherwise} \end{cases} \quad (39)$$

where \mathcal{N}_Y is the normal bundle of Y , viewed as a coherent sheaf on X . The connecting homomorphism

$$\mathcal{T}_X \longrightarrow \mathcal{E}xt^1(\mathcal{I}, \mathcal{I}) \cong \mathcal{N}_Y$$

is the natural projection of \mathcal{T}_X onto the normal bundle, which is surjective. We remark that, although (39) holds for an arbitrary local complete intersection, the surjectivity of the connecting homomorphism really requires Y to be smooth.

We conclude that the Atiyah algebroid $\mathcal{L} = \mathcal{L}(\mathcal{I})$ is quasi-isomorphic to its zeroth cohomology, i.e. \mathcal{L} is a coherent sheaf sitting purely in degree zero. But this sheaf is not a vector bundle; it has Tor amplitude $[-1, 0]$, so that it is an honest Lie 2-algebroid. From (38), we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{L} \longrightarrow \mathcal{T}_X(-\log Y) \longrightarrow 0$$

where $\mathcal{T}_X(-\log Y) \subset \mathcal{T}_X$ is the kernel of the projection $\mathcal{T}_X \rightarrow \mathcal{N}_Y$, i.e. the subsheaf of vector fields that are tangent to Y . Thus the orbits of \mathcal{L} are the connected components of Y and its complement. Since $\text{ch}_2(\mathcal{I})$ is, up to sign, the class $[Y]$ Poincaré dual to Y , we arrive at the

Theorem 5.13. *Let $Y \subset X$ be a smooth codimension-two subvariety in a complex manifold or smooth algebraic variety. Then there is a canonical twisted Courant algebroid on X whose twisting class is $-[Y] \in H^2(X, \Omega_X^{\geq 2})$, and whose orbits are the connected components of Y and its complement.*

This twisted Courant algebroid is always locally equivalent to a Courant algebroid, which we can describe concretely as follows. On a sufficiently small affine open subset $U \subset X$, we can find a flat rank-two vector bundle (\mathcal{V}, ∇) and a section $s \in H^0(U, \mathcal{V})$ whose zero scheme is $Y \cap U$. In this way we obtain the Koszul resolution

$$\mathcal{I}|_U \cong (\det \mathcal{V}^\vee \xrightarrow{s} \mathcal{V}^\vee),$$

which we may use to compute the derived endomorphisms $\mathbb{R}\mathcal{E}nd(\mathcal{I}|_U)$ and the Atiyah algebroid $\mathcal{L}|_U$.

In degree zero, we get $\mathfrak{g} = \mathbb{R}\mathcal{E}nd^0(\mathcal{I}|_U) = \mathcal{E}nd(\mathcal{V}) \oplus \mathcal{O}_U$ with the obvious Lie bracket, and with pairing given by the difference of the trace pairings. The connection on \mathcal{V} allows us to identify $\mathcal{L}|_U = \mathcal{T}_U \oplus \mathfrak{g}$ so that we get a transitive Courant algebroid

$$\mathcal{E}_0 = \mathcal{T}_U \oplus \mathfrak{g} \oplus \mathcal{T}_U^\vee,$$

as in Section 5.4.1.

Identifying the degree-one piece of $\mathcal{L}|_U$ with the bundle $\mathbb{R}\mathcal{E}nd^1(\mathcal{I}|_U) = \mathcal{V}$, the differential on $\mathcal{L}|_U$ is given by the map

$$\begin{aligned} \delta: \quad \mathcal{T}_U \oplus \mathfrak{g} &\rightarrow \mathcal{V} \\ (\xi, (\phi, f)) &\mapsto \nabla_\xi s + \phi s - f s \end{aligned}$$

for $\xi \in \mathcal{T}_U$, $(\phi, f) \in \mathfrak{g}$. Because s vanishes transversely, this map is surjective.

Now δ evidently extends to a surjection $\mathcal{E}_0 \rightarrow \mathcal{V}$ whose kernel $\mathcal{K} \subset \mathcal{E}_0$ is a coisotropic subbundle that is preserved by the Courant bracket. The annihilator $\mathcal{K}^\perp \subset \mathcal{K}$ is the image of \mathcal{V}^\vee under the dual map $\mathcal{V}^\vee \rightarrow \mathcal{E}^\vee \cong \mathcal{E}$. In this way, we obtain the desired Courant algebroid by coisotropic reduction:

$$\mathcal{E} = \mathcal{K}/\mathcal{K}^\perp = \mathcal{H}^0(\mathcal{V}^\vee \rightarrow \mathcal{E}_0 \rightarrow \mathcal{V})$$

which its induced bracket, anchor and pairing.

6 Two-shifted Lagrangians

Let \mathcal{L} be a two-shifted symplectic algebroid on X , and let \mathcal{E} be the corresponding twisted Courant algebroid. Although the projection map $X \rightarrow [X/\mathcal{L}]$ may be isotropic as above, it is essentially never Lagrangian. Indeed, we recall from [Example 4.8](#) that the Lagrangian condition forces $\mathcal{L} \cong \mathcal{T}_X^\vee[1]$, which means that $\mathcal{E} = 0$. Nevertheless, there may be many Lagrangians of the form

$$[Y/\mathcal{M}] \rightarrow [X/\mathcal{L}]$$

where $Y \subset X$ is a closed submanifold, and \mathcal{M} is an L_∞ algebroid on Y . In this section, we give a classification of such Lagrangians in terms of twisted Dirac structures in twisted Courant algebroids.

6.1 Twisted Dirac structures

Let X be a manifold, and let $f: Y \rightarrow X$ be the inclusion of a closed submanifold. Suppose that \mathcal{E} is a twisted Courant algebroid on X whose twisting cocycle lies in the relative de Rham complex $\Omega_{X,Y}^\bullet = \ker(\Omega_X^\bullet \rightarrow \Omega_Y^\bullet)$.

The restriction $f^*\mathcal{E}$ is a twisted vector bundle on Y equipped with a nondegenerate symmetric pairing, and so it makes sense to speak of twisted subbundles $\mathcal{F} \subset f^*\mathcal{E}$ that are isotropic or Lagrangian. Here, by a “twisted subbundle”, we mean that on any affine chart, \mathcal{F} is a subbundle of \mathcal{E} , and on the overlap of two charts, the subbundles are preserved by the transition functions of \mathcal{E} .

Applying the anchor to such a twisted subbundle, one obtains a subsheaf $a(\mathcal{F}) \subset f^*\mathcal{T}_X$. We say that \mathcal{F} is **compatible with the anchor** if $a(\mathcal{F}) \subset \mathcal{T}_Y$. In this case, it is easy to see that on any affine chart $U \subset X$ there is a well-defined bracket

$$[[-, -]]: \mathcal{F} \times \mathcal{F} \rightarrow f^*\mathcal{E}$$

defined by restriction of the Courant bracket on \mathcal{E} . In particular, it makes sense to ask if \mathcal{F} is **involutive**, i.e. $[[\mathcal{F}, \mathcal{F}]] \subset \mathcal{F}$. This allows us to extend the definition of a Dirac structure with support [\[3, 16, 58\]](#) to the twisted setting:

Definition 6.1. Let $f: Y \rightarrow X$ be an embedding of a closed submanifold. A **twisted Dirac pair on (X, Y)** is a pair $(\mathcal{E}, \mathcal{F})$ consisting of a twisted Courant algebroid \mathcal{E} whose twisting cocycle lies in $\Omega_{X,Y}^{\geq 2} \subset \Omega_X^{\geq 2}$, and a twisted Lagrangian subbundle $\mathcal{F} \subset f^*\mathcal{E}$ that is compatible with the anchor and involutive.

Twisted Dirac pairs are the objects of a natural strict 2-groupoid $\mathrm{TDir}(X, Y)$. For pairs $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{E}', \mathcal{F}')$ the morphisms are given by the subgroupoid

$$\mathrm{Hom}_{\mathrm{TDir}(X,Y)}((\mathcal{E}, \mathcal{F}), (\mathcal{E}', \mathcal{F}')) \subset \mathrm{Hom}_{\mathrm{TCA}(X)}(\mathcal{E}, \mathcal{E}')$$

consisting of the morphisms in TCA that preserve the twisted subbundles, and for which all form data lie in $\Omega_{X,Y}^{\geq 2} \subset \Omega_X^{\geq 2}$.

Remark 6.2. The natural forgetful map $\mathrm{TD}\mathrm{ir}(Y, X) \rightarrow \mathrm{TCA}(X)$ is not a fibration, so to define the space of Dirac structures in a fixed twisted Courant algebroid $\mathcal{E} \in \mathrm{TCA}(X)$, we must take its homotopy fibre in $\mathrm{TD}\mathrm{ir}(X, Y)$ instead of its strict fibre. For example, we should choose an isomorphism of \mathcal{E} with an equivalent model \mathcal{E}' for which the twisting cocycle lies in $\Omega_{X,Y}^2 \subset \Omega_X^2$. \square

If $(\mathcal{E}, \mathcal{F})$ is a Dirac pair on (X, Y) , the anchor gives a morphism

$$a^*: \mathcal{N}_Y^\vee \rightarrow \mathcal{F}.$$

Indeed, consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{N}_Y^\vee & \longrightarrow & f^* \mathcal{T}_X^\vee & \longrightarrow & \mathcal{T}_Y^\vee \longrightarrow 0 \\ & & \downarrow & & \downarrow a^* & & \downarrow a^\vee \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & f^* \mathcal{E} & \longrightarrow & \mathcal{F}^\vee \longrightarrow 0 \end{array}$$

The top sequence is exact by definition. The bottom sequence is exact since $\mathcal{F} \subset f^* \mathcal{E}$ is Lagrangian. Therefore, we get a unique morphism $\mathcal{N}_Y^\vee \rightarrow \mathcal{F}$ denoted by the dashed arrow.

The twisting class of the pair $(\mathcal{E}, \mathcal{F})$ is evidently a refinement of the twisting class $[\mathcal{E}] \in H^2(\Omega_X^{\geq 2})$ to a class in relative cohomology:

$$[\mathcal{E}, \mathcal{F}] \in H^2(\Omega_{X,Y}^{\geq 2}).$$

Notice that the exterior power of the exact sequence

$$0 \longrightarrow \mathcal{N}_Y^\vee \longrightarrow f^* \Omega_X^1 \longrightarrow \Omega_Y^1 \longrightarrow 0$$

gives rise to a natural projection $\Omega_{X,Y}^2 \rightarrow \Omega_Y^1 \otimes \mathcal{N}_Y^\vee$. The image of $[\mathcal{E}, \mathcal{F}]$ under the resulting map

$$H^2(\Omega_{X,Y}^{\geq 2}) \rightarrow H^2(Y, \Omega_Y^1 \otimes \mathcal{N}_Y^\vee)$$

measures the twisting of the transition functions of \mathcal{F} , as is evident from the formula (18) for the twisting of \mathcal{E} .

6.2 Classification of two-shifted Lagrangians

We now establish a classification of two-shifted Lagrangians. Suppose that $Y \subset X$ is a closed submanifold. We consider the ∞ -groupoid $\mathbf{Lag}_2(X, Y)$ whose objects are tuples consisting of a 2-shifted symplectic algebroid $(\mathcal{L}, \omega) \in \mathbf{SA}_2(X)$, an L_∞ -algebroid $\mathcal{M} \in \mathbf{LA}(Y)$, and a Lagrangian morphism $[Y/\mathcal{M}] \rightarrow [X/\mathcal{L}]$. Then we have the following:

Theorem 6.3. *The ∞ -groupoid $\mathbf{Lag}_2(X, Y)$ is equivalent to the strict 2-groupoid $\mathrm{TD}\mathrm{ir}(X, Y)$ of twisted Dirac pairs.*

The strategy of the proof is parallel to that of [Theorem 5.5](#). Once again, we fix an affine manifold U , this time with a closed submanifold $V \subset U$, and we consider a diagram of strict 2-groupoids

$$\begin{array}{ccc} & \text{TDir}_{\text{conn}}(U, V) & \\ \swarrow & & \searrow \sim \\ \text{Lag}_2(U, V) & & \text{TDir}(U, V) \end{array}$$

where the objects of $\text{TDir}_{\text{conn}}(U, V)$ are twisted Dirac pairs in which the twisted Courant algebroids is equipped with a metric connection and the Dirac structures is equipped with a tensor $\nu \in \wedge^2 \mathcal{F}^\vee \otimes \mathcal{N}_V^\vee$. We will construct a functor $\text{TDir}_{\text{conn}}(U, V) \rightarrow \text{Lag}_2(U, V)$ which we will prove is an equivalence.

6.2.1 Objects

We begin by constructing the equivalence on the level of objects. Using [Theorem 5.5](#) we identify

$$\mathcal{L} \cong \left(\mathcal{T}_U^\vee \longrightarrow \mathcal{E} \right)$$

for a twisted Courant algebroid \mathcal{E} on U , with symplectic form $\omega \in \Omega_{\text{cl}}^{2,2}([U/\mathcal{L}])$. It is then easy to see that a Lagrangian structure forces \mathcal{M} to be concentrated in degrees -1 and 0 .

An L_∞ -morphism $\mathcal{M} \rightarrow \mathcal{L}$ is given by morphisms

$$g: \mathcal{M} \rightarrow f^* \mathcal{L} \quad \quad \quad \tilde{g}: \wedge^2 \mathcal{M}_0 \rightarrow f^* \mathcal{T}_U^\vee$$

compatible with the anchor and satisfying (27)–(29).

An isotropic structure on $[V/\mathcal{M}] \rightarrow [U/\mathcal{L}]$ is given by elements

$$\tau \in \mathcal{M}_0^\vee \otimes \Omega^1(V) \quad \quad \quad H \in \Omega^3(V)$$

satisfying $(f, g)^* \omega = \delta_{\text{Tw}}(\tau + H)$, which gives the following equations analogous to (30)–(33):

$$f^* K = dH \quad (40)$$

$$f^* gu = -\tau \delta u \quad (41)$$

$$\langle gx, gy \rangle = \frac{1}{2}(\iota_{ax} \tau y + \iota_{ay} \tau x) \quad (42)$$

$$\begin{aligned} f^* \psi(gx, gy) - f^* \tilde{g}(x, y) &= \tau[x, y] - \mathcal{L}_{ax} \tau y + \mathcal{L}_{ay} \tau x \\ &\quad + \frac{1}{2} d(\iota_{ax} \tau y - \iota_{ay} \tau x) + \iota_{ax} \iota_{ay} H. \end{aligned} \quad (43)$$

for $x, y \in \mathcal{M}_0$ and $u \in \mathcal{M}_1$. We will now show that one can rectify the Lagrangian $[V/\mathcal{M}] \rightarrow [U/\mathcal{L}]$.

Proposition 6.4. *Let \mathcal{L} be a two-shifted symplectic L_∞ algebroid on X corresponding to a twisted Courant algebroid \mathcal{E} and let $[Y/\mathcal{M}] \rightarrow [X/\mathcal{L}]$ a Lagrangian. Then \mathcal{M} is quasi-isomorphic to a subcomplex*

$$\mathcal{M} \cong \left(\mathcal{N}_V^\vee \longrightarrow \mathcal{F} \right) \subset \left(f^* \mathcal{T}_U^\vee \longrightarrow f^* \mathcal{E} \right)$$

where $\mathcal{F} \subset f^*\mathcal{E}$ is a Lagrangian subbundle, and the symplectic structure vanishes identically on \mathcal{M} .

Proof. We consider the complex $\widetilde{\mathcal{M}} = \mathcal{M} \oplus f^*\mathcal{T}_U^\vee \oplus f^*\mathcal{T}_U^\vee[1]$ with differential

$$f^*\mathcal{T}_U^\vee \oplus \mathcal{M}_1 \xrightarrow{\begin{pmatrix} \text{id} & g \\ 0 & \delta \end{pmatrix}} f^*\mathcal{T}_U^\vee \oplus \mathcal{M}_0$$

and the complex $\widetilde{\mathcal{F}} = \mathcal{M} \oplus f^*\mathcal{T}_U^\vee$ with the differential twisted by g .

As in the proof of [Proposition 5.6](#) we get a deformation retract of the form $p: \widetilde{\mathcal{M}} \rightrightarrows \mathcal{M}: i$, so we can transfer the L_∞ algebroid structure from \mathcal{M} to $\widetilde{\mathcal{M}}$. The Lagrangian structure on $\widetilde{\mathcal{M}}$ takes the following shape:

$$\begin{array}{ccccc} \pi^*\mathcal{T}_{[V/\widetilde{\mathcal{M}}]} & & f^*\mathcal{T}_U^\vee \oplus \mathcal{M}_1 & \longrightarrow & f^*\mathcal{T}_U^\vee \oplus \mathcal{M}_0 & \longrightarrow & \mathcal{T}_V \\ \downarrow & & \downarrow (0,g) & \nearrow & \downarrow (0,g) & \nearrow & \downarrow \\ f^*\pi^*\mathcal{T}_{[U/\mathcal{L}]} & & f^*\mathcal{T}_U^\vee & \longrightarrow & f^*\mathcal{E} & \longrightarrow & f^*\mathcal{T}_U \\ \downarrow \omega & & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\ \pi^*\mathcal{T}_{[V/\widetilde{\mathcal{M}}]}[2] & & \mathcal{T}_V^\vee & \longrightarrow & f^*\mathcal{T}_U \oplus \mathcal{M}_0^\vee & \longrightarrow & f^*\mathcal{T}_U \oplus \mathcal{M}_1^\vee \end{array}$$

with the null homotopy of the composite given by $\tau \in \mathcal{M}_0^\vee \otimes \mathcal{T}_V^\vee$.

The identity operator on \mathcal{T}_U^\vee gives a projection $\widetilde{\mathcal{M}}_0 \rightarrow f^*\mathcal{T}_U^\vee$, which we use as a homotopy to modify the map $\widetilde{\mathcal{M}} \rightarrow \mathcal{M} \rightarrow f^*\mathcal{L}$. Using the formulae in [Section 5.2.3](#), we see that the modified Lagrangian structure has the form

$$\begin{array}{ccccc} f^*\mathcal{T}_U^\vee \oplus \mathcal{M}_1 & \longrightarrow & f^*\mathcal{T}_U^\vee \oplus \mathcal{M}_0 & \longrightarrow & \mathcal{T}_V^\vee \\ \downarrow (\text{id}, 0) & \nearrow & \downarrow (a^\vee, g) & \nearrow & \downarrow \\ f^*\mathcal{T}_U^\vee & \longrightarrow & f^*\mathcal{E} & \longrightarrow & f^*\mathcal{T}_U^\vee \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\ \mathcal{T}_V^\vee & \longrightarrow & f^*\mathcal{T}_U \oplus \mathcal{M}_0^\vee & \longrightarrow & f^*\mathcal{T}_U \oplus \mathcal{M}_1^\vee \end{array}$$

with the null homotopy now given by the inclusion $\mathcal{T}_V \subset f^*\mathcal{T}_U$ and its dual.

The nondegeneracy condition on the Lagrangian $[V/\mathcal{M}] \rightarrow [U/\mathcal{L}]$ now implies that we have a self-dual exact triangle

$$\widetilde{\mathcal{F}} \longrightarrow \mathcal{T}_V \oplus f^*\mathcal{E} \oplus \mathcal{T}_V^\vee \longrightarrow \widetilde{\mathcal{F}}^\vee \longrightarrow \widetilde{\mathcal{F}}[1]$$

Since $\mathcal{T}_V \oplus f^*\mathcal{E} \oplus \mathcal{T}_V^\vee$ is concentrated in degree zero, and $\widetilde{\mathcal{F}}$ is concentrated in nonpositive degrees, we conclude that the projection $\widetilde{\mathcal{F}} \rightarrow \mathcal{H}^0(\widetilde{\mathcal{F}})$ is a quasi-isomorphism. This allows us to replace the L_∞ algebroid $\widetilde{\mathcal{M}}$ by $f^*\mathcal{T}_U^\vee \rightarrow \mathcal{H}^0(\widetilde{\mathcal{F}})$,

giving a Lagrangian structure of the form

$$\begin{array}{ccccc}
f^*\mathcal{T}_U^\vee & \longrightarrow & \mathcal{H}^0(\tilde{\mathcal{F}}) & \longrightarrow & \mathcal{T}_V \\
\downarrow \text{id} & & \downarrow & & \downarrow \\
f^*\mathcal{T}_U^\vee & \longrightarrow & f^*\mathcal{E} & \longrightarrow & f^*\mathcal{T}_U \\
\downarrow & & \downarrow & & \downarrow \text{id} \\
\mathcal{T}_V^\vee & \longrightarrow & \mathcal{H}^0(\tilde{\mathcal{F}})^\vee & \longrightarrow & f^*\mathcal{T}_U
\end{array}$$

with homotopy $\tau \in \mathcal{H}^0(\tilde{\mathcal{F}})^\vee \otimes \mathcal{T}_V^\vee$. From equation (41) we see that τ is surjective, and hence we may define a bundle \mathcal{F} by the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{N}_V^\vee & \longrightarrow & f^*\mathcal{T}_U^\vee & \longrightarrow & \mathcal{T}_V^\vee \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \text{id} \\
0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{H}^0(\tilde{\mathcal{F}}) & \xrightarrow{\tau} & \mathcal{T}_V^\vee \longrightarrow 0
\end{array}$$

with exact rows. Considering the columns as morphisms of two-term complexes, we obtain the desired quasi-isomorphism $\mathcal{M} \cong (\mathcal{N}_V^\vee \rightarrow \mathcal{F})$. \square

We now assume that $\mathcal{M} \cong (\mathcal{N}_V^\vee \rightarrow \mathcal{F})$ as in the Proposition, so that the morphism $g: \mathcal{M} \rightarrow f^*\mathcal{L}$ is the inclusion. The space of pairs of a closed form on $[U/\mathcal{L}]$ and an isotropic structure on $[V/\mathcal{M}] \rightarrow [U/\mathcal{L}]$ is given by the homotopy fibre of the projection $\Omega^\bullet([U/\mathcal{L}]) \rightarrow \Omega^\bullet([V/\mathcal{M}])$. Since the projection is now surjective, this is just the kernel. Thus we may assume that all the form data on $[V/\mathcal{M}]$, namely τ and H , are zero.

The equations (27), (28) and (29) for an L_∞ morphism now uniquely determine, the binary bracket $\mathcal{M}_0 \times \mathcal{M}_0 \rightarrow \mathcal{M}_0$, the binary bracket $\mathcal{M}_0 \times \mathcal{M}_1 \rightarrow \mathcal{M}_1$, and the triple bracket $\wedge^3 \mathcal{M}_0 \rightarrow \mathcal{M}_1$, respectively. The isotropy condition (43) then implies that

$$\tilde{g}(x, y) = \psi(x, y) + \nu(x, y)$$

for some $\nu \in \wedge^2 \mathcal{F}^\vee \otimes f^*\mathcal{N}_V^\vee$. Hence by the defining relation (26) between the Courant bracket and the binary bracket on \mathcal{L} , the equations for the isotropic structure reduce to the single condition

$$\begin{aligned}
[x, y]_{\mathcal{M}} &= [x, y]_{\mathcal{L}} - \frac{1}{2} a_{\mathcal{E}}^* \tilde{g}(x, y) \\
&= \llbracket x, y \rrbracket - \frac{1}{2} a_{\mathcal{E}}^* \mathrm{d} \langle x, y \rangle - \frac{1}{2} a_{\mathcal{F}}^* \nu(x, y),
\end{aligned}$$

where the expressions $\llbracket x, y \rrbracket$ and $\mathrm{d} \langle x, y \rangle$ are defined by extending x and y to sections of \mathcal{E} in a neighbourhood of V and then restricting the results. Since $\mathcal{F} \subset f^*\mathcal{E}$ is isotropic, the expression $\mathrm{d} \langle x, y \rangle$ automatically lies in \mathcal{N}_V , which implies that $\llbracket x, y \rrbracket \in \mathcal{F}$, so that \mathcal{F} is a Dirac structure in \mathcal{E} . This gives the equivalence of $\mathrm{TDir}_{\mathrm{conn}}(U, V)$ and $\mathrm{Lag}_2(U, V)$ at the level of objects.

6.2.2 1-morphisms

The one-morphisms in $\mathbf{Lag}_2(U, V)$ are given by homotopy commutative diagrams of the form

$$\begin{array}{ccc} [V/\mathcal{M}] & \xrightarrow{\mu} & [V/\mathcal{M}'] \\ \downarrow & \swarrow h & \downarrow \\ [U/\mathcal{L}] & \xrightarrow{g} & [U/\mathcal{L}'] \end{array}$$

where the vertical morphisms are two-shifted Lagrangians defined by twisted Dirac pairs.

An L_∞ morphism $\mathcal{M} \rightarrow \mathcal{M}'$ consists of a chain morphism $\mu: \mathcal{M} \rightarrow \mathcal{M}'$ and a linear map $\tilde{\mu}: \wedge^2 \mathcal{F} \rightarrow \mathcal{N}_V^\vee$ satisfying (27)–(29) with g replaced by μ . Meanwhile by Theorem 5.5, the 1-morphism $[U/\mathcal{L}] \rightarrow [U/\mathcal{L}']$ is determined by a morphism of twisted Courant algebroids, consisting of a bundle map $g: \mathcal{E} \rightarrow \mathcal{E}'$ and a three-form $H \in \Omega^3(U)$. Finally, we have the data of a homotopy between the composites

$$\left(\mathcal{M} \xrightarrow{\mu} \mathcal{M}' \longrightarrow f^* \mathcal{L}' \right) \sim_h \left(\mathcal{M} \longrightarrow f^* \mathcal{L} \xrightarrow{g} f^* \mathcal{L}' \right),$$

compatible with the form data. It is determined by a bundle map $h: \mathcal{F} \rightarrow f^* \mathcal{T}_X^\vee$ that satisfies the following variants of (35)–(37):

$$\mu x - gx = -\frac{1}{2} a^\vee h \quad (44)$$

$$\mu u - u = -\frac{1}{2} h a^\vee u \quad (45)$$

$$\begin{aligned} h[x, y] - [hx, y] - [x, hy] &= \psi'(\mu x, \mu y) - \nu'(\mu x, \mu y) \\ &\quad + \tilde{\mu}(x, y) - \tilde{g}(x, y) \\ &\quad - \psi(x, y) + \nu(x, y) \end{aligned} \quad (46)$$

Since we assume the Lagrangian structure to be strict, we get $f^* H = 0$ and $f^* h x = 0$. In particular, $h \in \mathcal{F}^\vee \otimes \mathcal{N}_V^\vee$.

Consider the case when $h = 0$. Then equation (44) holds if and only if g intertwines the subbundle $\mathcal{F} \subset f^* \mathcal{E}$ and $\mathcal{F}' \subset f^* \mathcal{E}'$, and equation (45) means that μ in degree -1 is the identity. Equation (46) uniquely determines $\tilde{\mu}(x, y)$. It is not difficult to check that then the equations (27)–(29) for μ to be an L_∞ morphism are automatically satisfied. Thus we see that there is an inclusion $\mathbf{TDir}_{\text{conn}}(U, V) \rightarrow \mathbf{Lag}_2(U, V)$ at the level of one-morphisms.

6.2.3 2-morphisms

2-morphisms in $\mathbf{Lag}_2(U, V)$ are given by diagrams of the form

$$\begin{array}{ccc}
 [V/\mathcal{M}] & \xrightarrow{\mu_1} & [V/\mathcal{M}'] \\
 \downarrow & \Downarrow \chi & \downarrow \\
 & \xrightarrow{\mu_2} & \\
 [U/\mathcal{L}] & \xrightarrow{g_1} & [U/\mathcal{L}'] \\
 & \Downarrow & \\
 & \xrightarrow{g_2} &
 \end{array}$$

together with a homotopy $\mu_1 \overset{\chi}{\sim} \mu_2$ represented by morphism $\chi: \mathcal{F} \rightarrow \mathcal{N}_V^\vee$, and a 2-morphism $(g_1, H_1) \sim (g_2, H_2)$ represented by a two-form $B \in \Omega^2(U)$ such that $f^*B = 0$.

The homotopies satisfies the following equations:

$$\mu_2 x - \mu_1 x = -\frac{1}{2} a^\vee \chi x \quad (47)$$

$$\mu_2 u - \mu_1 u = -\frac{1}{2} \chi a^\vee x \quad (48)$$

$$\tilde{\mu}_2(x, y) - \tilde{\mu}_1(x, y) = \chi[x, y] - [\chi x, \mu_1 y] - [\mu_1 x, \chi y] \quad (49)$$

$$h_2 x - h_1 x = \chi x \quad (50)$$

Equation (48) is automatic in view of equations (50) and (45). Similarly, equation (49) follows from equations (50) and (46). Thus (50) implies that every 1-morphism is equivalent to one for which $h = 0$. Restricting to such 1-morphisms, we see that $\chi = 0$, and hence $\mu_2 = \mu_1$.

In this way we have produced a 2-functor $\mathbf{TDir}_{\text{conn}}(U, V) \rightarrow \mathbf{SA}_2(U, V)$ which is clearly fully faithful and essentially surjective by [Proposition 6.4](#).

6.3 Untwisted Dirac structures

The more classical notion of a Dirac structure concerns Lagrangian subbundles in an untwisted Courant algebroid. More specifically, we have a 1-groupoid $\mathbf{Dir}(X, Y)$ as follows:

Objects of $\mathbf{Dir}(X, Y)$ are given by Dirac pairs $(\mathcal{E}, \mathcal{F})$ for which \mathcal{E} is an (untwisted) Courant algebroid.

1-morphisms $(\mathcal{E}, \mathcal{F}) \rightarrow (\mathcal{E}', \mathcal{F}')$ in $\mathbf{Dir}(X, Y)$ are given by Courant algebroid isomorphisms $g: \mathcal{E} \rightarrow \mathcal{E}'$ that preserve the Dirac structures.

The symplectic counterpart is as follows. Consider a two-shifted isotropic structure $X \rightarrow [X/\mathcal{L}]$ representing an untwisted Courant algebroid \mathcal{E} . Then a Dirac structure in \mathcal{E} is the data of a Lagrangian $[Y/\mathcal{M}] \rightarrow [X/\mathcal{L}]$, forming a

commutative square

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ [Y/\mathcal{M}] & \longrightarrow & [X/\mathcal{L}], \end{array}$$

together with a homotopy of the isotropic structures on $Y \rightarrow [X/\mathcal{L}]$ induced by the two compositions. Such a diagram is an example of an **isotropic correspondence**. We denote by $\mathbf{Lag}_2^{\text{iso}}(Y, X)$ the ∞ -groupoid parametrizing such structures and their homotopies. An argument identical to the proof of [Proposition 5.8](#) then gives the

Proposition 6.5. *Let $Y \subset X$ be a closed submanifold. Then the ∞ -groupoid $\mathbf{Lag}_2^{\text{iso}}(Y, X)$ is equivalent to the 1-groupoid $\mathbf{Dir}(X, Y)$ of untwisted Dirac pairs.*

Consider now the special case in which $\mathcal{L} = \mathcal{T}_X$ and $\mathcal{M} = \mathcal{T}_Y$, so that we have an isotropic correspondence

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ [Y/\mathcal{T}_Y] & \longrightarrow & [X/\mathcal{T}_X] \end{array}$$

On the one hand, all form data on $[Y/\mathcal{T}_Y]$ and $[X/\mathcal{T}_X]$ are equivalent to zero, so that the isotropic structure on X is a 1-cocycle in $\Omega_X^{\geq 2}$, and the isotropic correspondence is given by a trivialization of its pullback to Y .

On the other hand, the isotropic structure $X \rightarrow [X/\mathcal{T}_X]$ corresponds to an exact Courant algebroid \mathcal{E} on X as in [Section 5.3](#). It pulls back to an isotropic structure $Y \rightarrow [Y/\mathcal{T}_Y]$, giving an exact Courant algebroid $f^!\mathcal{E}$ by the usual restriction formula

$$f^!\mathcal{E} = a^{-1}(\mathcal{T}_Y)/\mathcal{N}_Y^\vee.$$

The isotropic correspondence then results in a trivialization

$$f^!\mathcal{E} \cong \mathcal{T}_Y \oplus \mathcal{T}_Y^\vee$$

Under this isomorphism, the Lagrangian $[Y/\mathcal{T}_Y] \rightarrow [X/\mathcal{T}_X]$ corresponds to the Dirac structure $\mathcal{F} \subset f^*\mathcal{E}$ given by the preimage of \mathcal{T}_Y along the projection $a^{-1}(\mathcal{T}_Y) \rightarrow f^!\mathcal{E}$. It sits in an exact sequence

$$0 \longrightarrow \mathcal{N}_Y^\vee \longrightarrow \mathcal{F} \longrightarrow \mathcal{T}_Y \longrightarrow 0$$

and is referred to in [\[34, Section 6\]](#) as the **generalized tangent bundle** of the Courant trivialization. We arrive at the following Lagrangian analogue of [Corollary 5.10](#):

Corollary 6.6. *Let \mathcal{E} be an exact Courant algebroid on X . The set of generalized tangent bundles for $Y \subset X$ is in bijection with the set of Courant trivializations of $f^!\mathcal{E}$. These sets are torsors for the space $H^0(Y, \Omega_{\text{cl}}^2)$ of global closed two-forms on Y .*

7 One-shifted symplectic forms

7.1 Symplectic structures and exact Dirac pairs

The classification of one-shifted symplectic algebroids is a simple consequence of the results so far.

Indeed, recall from [Example 4.7](#), that one-shifted symplectic structures on $[X/\mathcal{L}]$ are equivalent to Lagrangian structures on the projection $[X/\mathcal{L}] \rightarrow \{*\}$, where $\{*\}$ denotes a point equipped with the trivial two-shifted symplectic structure. Note moreover that we can factor the projection via the anchor map $\mathcal{L} \rightarrow \mathcal{T}_X$ as follows:

$$\begin{array}{ccc} & & [X/\mathcal{T}_X] \\ & \nearrow \text{dashed} & \downarrow \\ [X/\mathcal{L}] & \longrightarrow & \{*\} \end{array}$$

Since the vertical arrow is an étale map (a special case of the construction in [Section 4.2.2](#)), we may equivalently view a one-shifted symplectic structure on $[X/\mathcal{L}]$ as a Lagrangian structure on the map $[X/\mathcal{L}] \rightarrow [X/\mathcal{T}_X]$, where the latter is equipped with the trivial two-shifted symplectic structure. (We thank the referee for pointing out that our argument in a previous version of the paper could be rephrased in the present, more conceptual, fashion.)

Now a Lagrangian structure on $[X/\mathcal{L}] \rightarrow [X/\mathcal{T}_X]$ automatically induces an isotropic structure on $X \rightarrow [X/\mathcal{T}_X]$ by pullback. By [Corollary 5.10](#), this gives an exact Courant algebroid \mathcal{E} on X , and the Lagrangian structure embeds \mathcal{L} as a Dirac structure in \mathcal{E} . We call such pairs $(\mathcal{E}, \mathcal{L})$ **exact Dirac pairs**; they form a full subgroupoid $\text{ExDir}(X) \subset \text{Dir}(X, X)$. We therefore recover the infinitesimal characterization [\[15, 69\]](#) of quasi-symplectic groupoids:

Theorem 7.1. *For any manifold X , there is a canonical equivalence,*

$$\text{SA}_1(X) \cong \text{ExDir}(X)$$

between the ∞ -groupoid of 1-shifted symplectic algebroids on X , and the 1-groupoid of exact Dirac pairs.

Remark 7.2. On an affine manifold U , we can choose a splitting of the exact sequence

$$0 \longrightarrow \mathcal{T}_U^\vee \longrightarrow \mathcal{E} \longrightarrow \mathcal{T}_U \longrightarrow 0,$$

giving an identification of \mathcal{E} with the standard Courant algebroid $\mathcal{T}_U \oplus \mathcal{T}_U^\vee$, but with the bracket twisted by a closed three-form $H \in \Omega_{\text{cl}}^3(U)$ as in [Example 5.4](#). In this presentation, the $(2, 1)$ -form underlying the shifted symplectic structure on $[X/\mathcal{L}]$ is completely determined by the following quasi-isomorphism of complexes on X , associated with the embedding of \mathcal{L} as a Lagrangian subbundle in

$$\mathcal{E} \cong \mathcal{T}_U \oplus \mathcal{T}_U^\vee:$$

$$\begin{array}{ccc} \pi^* \mathcal{T}_{[U/\mathcal{L}]} & & \mathcal{L} \longrightarrow \mathcal{T}_U \\ \downarrow & & \downarrow \\ \pi^* \mathcal{T}_{[U/\mathcal{L}]}^\vee[1] & & \mathcal{T}_U^\vee \longrightarrow \mathcal{L}^\vee \end{array}$$

Meanwhile, the closure data is provided by the three-form H that modifies the standard Courant bracket. In the non-affine setting, \mathcal{E} need not split, so such a description of the symplectic structure may only exist locally. \square

There is also a symplectic interpretation of the “tensor product” of suitably transverse exact Dirac structures, introduced in [2, 34]. Indeed, let $(\mathcal{E}_1, \mathcal{L}_1)$ and $(\mathcal{E}_2, \mathcal{L}_2)$ be exact Dirac pairs. Then we have a pair of two-shifted Lagrangian morphisms $[X/\mathcal{L}_i] \rightarrow [X/\mathcal{T}_X]$. Let us denote by $[X/\mathcal{L}_2]$ the Lagrangian in which the signs of all form data are reversed. Then we can define a new Lie algebroid $\mathcal{L}_1 \boxtimes \mathcal{L}_2$ by taking the fibre product

$$[X/\mathcal{L}_1] \times_{[X/\mathcal{T}_X]} \overline{[X/\mathcal{L}_2]} \cong [X/(\mathcal{L}_1 \boxtimes \mathcal{L}_2)].$$

Since the fibre product of two n -shifted Lagrangians is always $(n-1)$ -shifted symplectic [54], we see that $[X/(\mathcal{L}_1 \boxtimes \mathcal{L}_2)]$ carries a canonical one-shifted symplectic structure, and hence we obtain a new exact Dirac pair $(\mathcal{E}_1 \boxtimes \mathcal{E}_2, \mathcal{L}_1 \boxtimes \mathcal{L}_2)$.

To see that this one-shifted symplectic structure coincides with the construction in [2, 34], we note that the underlying isotropic structures are additive under \boxtimes and hence the Ševera class of the exact Courant algebroid is additive under this operation, which determined $\mathcal{E}_1 \boxtimes \mathcal{E}_2$ up to isomorphism; see [34, p. 88] for the functorial construction. Considering the corresponding fibre product on tangent complexes, we immediately see that

$$\mathcal{L}_1 \boxtimes \mathcal{L}_2 \cong \mathcal{L}_1 \times_{\mathcal{T}_X} \mathcal{L}_2 \cong \left(\mathcal{L}_1 \oplus \mathcal{L}_2 \longrightarrow \mathcal{T}_X \right)$$

where $\mathcal{L}_1 \oplus \mathcal{L}_2$ sits in degree zero. When the anchor maps are transverse, this complex is quasi-isomorphic to its zeroth cohomology, which is the usual fibre product of vector bundles, giving the desired formula for the Dirac structure. When the anchors are not transverse, one could still make sense of the tensor product as some derived version of a Dirac structure, which would have nontrivial cohomology in degree one.

To see that \boxtimes is monoidal, notice that the associativity of Lagrangian fibre products implies the associativity relation $(\mathcal{L}_1 \boxtimes \mathcal{L}_2) \boxtimes \mathcal{L}_3 \cong \mathcal{L}_1 \boxtimes (\mathcal{L}_2 \boxtimes \mathcal{L}_3)$ on exact Dirac structures. The monoidal unit is given by the identity Lagrangian $[X/\mathcal{T}_X] \rightarrow [X/\mathcal{T}_X]$, which corresponds to the Dirac structure $\mathcal{T}_X \subset \mathcal{T}_X \oplus \mathcal{T}_X^\vee$.

7.2 Lagrangians with support and Courant trivializations

Now suppose that X is a manifold, and $(\mathcal{E}, \mathcal{L})$ is an exact Dirac pair, defining a one-shifted symplectic structure on $[X/\mathcal{L}]$. We now classify Lagrangians of the form $[Y/\mathcal{M}] \rightarrow [X/\mathcal{L}]$, where $f: Y \rightarrow X$ is a closed submanifold.

We view the symplectic structure on $[X/\mathcal{L}]$ as a Lagrangian $[X/\mathcal{L}] \rightarrow [X/\mathcal{T}_X]$ as above, and consider the commutative diagram

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ [Y/\mathcal{M}] & \longrightarrow & [X/\mathcal{L}] \\ \downarrow & & \downarrow \\ [Y/\mathcal{T}_Y] & \longrightarrow & [X/\mathcal{T}_X] \end{array}$$

The outermost rectangle is then an isotropic correspondence of the type considered in [Section 6.3](#), and so we obtain a Courant trivialization on Y . Let $\mathcal{F} \subset f^*\mathcal{E}$ be the corresponding generalized tangent bundle. Then the tangent complexes of the bottom square give the diagram

$$\begin{array}{ccc} (\mathcal{M} \rightarrow \mathcal{T}_Y) & \longrightarrow & (f^*\mathcal{L} \rightarrow f^*\mathcal{T}_X) \\ \downarrow & & \downarrow \\ (\mathcal{N}_Y^\vee \rightarrow \mathcal{F} \rightarrow \mathcal{T}_Y) & \longrightarrow & (f^*\mathcal{T}_X^\vee \rightarrow f^*\mathcal{E} \rightarrow f^*\mathcal{T}_X) \end{array}$$

and we immediately see that \mathcal{M} maps to the intersection $f^*\mathcal{L} \cap \mathcal{F} \subset f^*\mathcal{E}$.

Definition 7.3. Let $(\mathcal{E}, \mathcal{L})$ be an exact Dirac pair. A Courant trivialization $f^!\mathcal{E} \cong \mathcal{T}_Y \oplus \mathcal{T}_Y^\vee$ is **compatible with \mathcal{L}** if the subsheaf $\mathcal{F} \cap \mathcal{L} \subset f^*\mathcal{E}$ is actually an embedded subbundle.

We claim that the Lagrangian condition is equivalent to the induced map $\mathcal{M} \rightarrow f^*\mathcal{L} \cap \mathcal{F}$ being an isomorphism, so that the data of a Lagrangian is the same as the data of a compatible Courant trivialization. To see this, consider the intersection

$$W = [X/\mathcal{L}] \times_{[X/\mathcal{T}_X]} [Y/\mathcal{T}_Y]$$

equipped with its 1-shifted symplectic structure, and observe that $[Y/\mathcal{M}] \rightarrow [X/\mathcal{L}]$ is Lagrangian if and only if $[Y/\mathcal{M}] \rightarrow W$ is. Using the Lagrangian condition on $[X/\mathcal{L}] \rightarrow [X/\mathcal{T}_X]$, we may identify the tangent complex of the fibre product with the homotopy kernel of the morphism $\mathcal{T}_{[Y/\mathcal{T}_Y]} \rightarrow \mathcal{T}_{[X/\mathcal{L}]}^\vee[2]$ defined by the symplectic form. We thus obtain an equivalence

$$\begin{aligned} \pi^*\mathcal{T}_W &\cong \left(\mathcal{N}_Y^\vee \longrightarrow \mathcal{F} \oplus f^*\mathcal{T}_X^\vee \longrightarrow f^*\mathcal{L}^\vee \oplus \mathcal{T}_Y \right) \\ &\cong \left(\mathcal{F} \oplus \mathcal{T}_Y^\vee \longrightarrow f^*\mathcal{L}^\vee \oplus \mathcal{T}_Y \right) \end{aligned}$$

where the component $\mathcal{F} \rightarrow f^*\mathcal{L}^\vee$ is induced by the nondegenerate pairing on \mathcal{E} . The Lagrangian condition is now equivalent to having an exact sequence

$$(\mathcal{M} \longrightarrow \mathcal{T}_Y) \rightarrow \left(\mathcal{F} \oplus \mathcal{T}_Y^\vee \longrightarrow f^*\mathcal{L}^\vee \oplus \mathcal{T}_Y \right) \rightarrow \left(\mathcal{T}_Y^\vee \longrightarrow \mathcal{M}^\vee \right),$$

which in turn is equivalent to the exactness of the complex

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{F} \longrightarrow f^*\mathcal{L}^\vee \longrightarrow \mathcal{M}^\vee \longrightarrow 0$$

Using the fact that \mathcal{F} and \mathcal{L} are Lagrangian subbundles, this is equivalent to having $\mathcal{M} = f^*\mathcal{L} \cap \mathcal{F}$, as claimed. We have therefore arrived at the following

Theorem 7.4. *The ∞ -groupoid $\mathbf{Lag}_1(X, Y)$ consisting of one-shifted Lagrangian morphisms $[Y/\mathcal{M}] \rightarrow [X/\mathcal{L}]$ is equivalent to the 1-groupoid of exact Dirac pairs equipped with a compatible Courant trivialization along Y .*

Example 7.5. Suppose that $\mathcal{E} = \mathcal{T}_X \oplus \mathcal{T}_X^\vee$ is the standard Courant algebroid and $\mathcal{L} = \mathcal{A} \oplus \mathcal{A}^\perp \subset \mathcal{E}$ where $\mathcal{A} \subset \mathcal{T}_X$ is the involutive subbundle determined by a regular foliation of X and $\mathcal{A}^\perp \subset \mathcal{T}_X^\vee$ is its annihilator. If $Y \subset X$ is a union of leaves of the foliation, then the canonical trivialization $f^!\mathcal{E} \cong \mathcal{T}_Y \oplus \mathcal{T}_Y^\vee$ is compatible with \mathcal{L} . Indeed, we have the generalized tangent bundle

$$\mathcal{F} = \mathcal{T}_Y \oplus \mathcal{N}_Y^\vee$$

so that

$$\mathcal{F} \cap f^*\mathcal{L} = f^*\mathcal{A} \oplus \mathcal{N}_Y^\vee$$

which is evidently a subbundle. \square

7.3 Isotropic and Lagrangian quotients

Consider now the case of a Lagrangian of the form

$$X \rightarrow [X/\mathcal{L}],$$

so that the Lie algebroid on the source is trivial. Thus an isotropic structure is simply a Courant trivialization of \mathcal{E} on X , embedding \mathcal{L} as a Dirac structure in the standard Courant algebroid $\mathcal{T}_X \oplus \mathcal{T}_X^\vee$. According to [Example 4.8](#), the Lagrangian condition then forces the projection $\mathcal{L} \rightarrow \mathcal{T}_X^\vee$ to be an isomorphism; as is well known, such Dirac structures are precisely the graphs of Poisson bivectors.

Corollary 7.6. *Let X be a manifold. The ∞ -groupoid $\mathbf{SA}_1^{\text{iso}}(X)$ of one-shifted isotropic quotients $X \rightarrow [X/\mathcal{L}]$ is equivalent to the discrete set of Dirac structures in the standard Courant algebroid $\mathcal{T}_X \oplus \mathcal{T}_X^\vee$. Under this equivalence, the subgroupoid of Lagrangians is identified with the set of Poisson structures on X .*

The equivalence between Poisson structures and one-shifted Lagrangians is closely related to the correspondence between Poisson structures and symplectic groupoids [\[41, 68\]](#). Indeed, the quotient map $X \rightarrow [X/\mathcal{L}]$ gives an atlas for the stack $[X/\mathcal{L}]$, via the formal groupoid

$$G = X \times_{[X/\mathcal{L}]} X \rightrightarrows X$$

integrating the Lie algebroid \mathcal{L} . Being the fibre product of Lagrangians in a one-shifted symplectic stack, G carries a canonical zero-shifted symplectic structure, and this form is multiplicative by construction.

7.4 Lagrangian correspondences

We close with the following interpretation of the relation between coisotropic submanifolds and Lagrangian subgroupoids [19]:

Proposition 7.7. *For a closed submanifold $f: Y \rightarrow X$, the ∞ -groupoid of one-shifted Lagrangian correspondences*

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ [Y/\mathcal{M}] & \longrightarrow & [X/\mathcal{L}] \end{array}$$

is equivalent to the discrete set of Poisson structures on X for which Y is a coisotropic submanifold.

Proof. Computing the tangent complex of the fibre product, we immediately see that the Lagrangian condition on the map

$$Y \rightarrow [Y/\mathcal{M}] \times_{[X/\mathcal{L}]} X$$

is equivalent to the exactness of the sequence

$$\mathcal{T}_Y \rightarrow (\mathcal{M} \rightarrow \mathcal{T}_Y \oplus f^*\mathcal{T}_X \oplus \mathcal{L} \rightarrow f^*\mathcal{T}_X) \rightarrow \mathcal{T}_Y^\vee.$$

The middle complex is quasi-isomorphic to $(\mathcal{M} \rightarrow \mathcal{T}_Y \oplus \mathcal{L})$, and using $\mathcal{L} \cong \mathcal{T}_X^\vee$ we get an isomorphism $\mathcal{M} \cong \mathcal{N}_Y^\vee$, with anchor map induced by the Poisson bivector. Thus Y is coisotropic. \square

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